Combinatoire de Catalan rectangulaire

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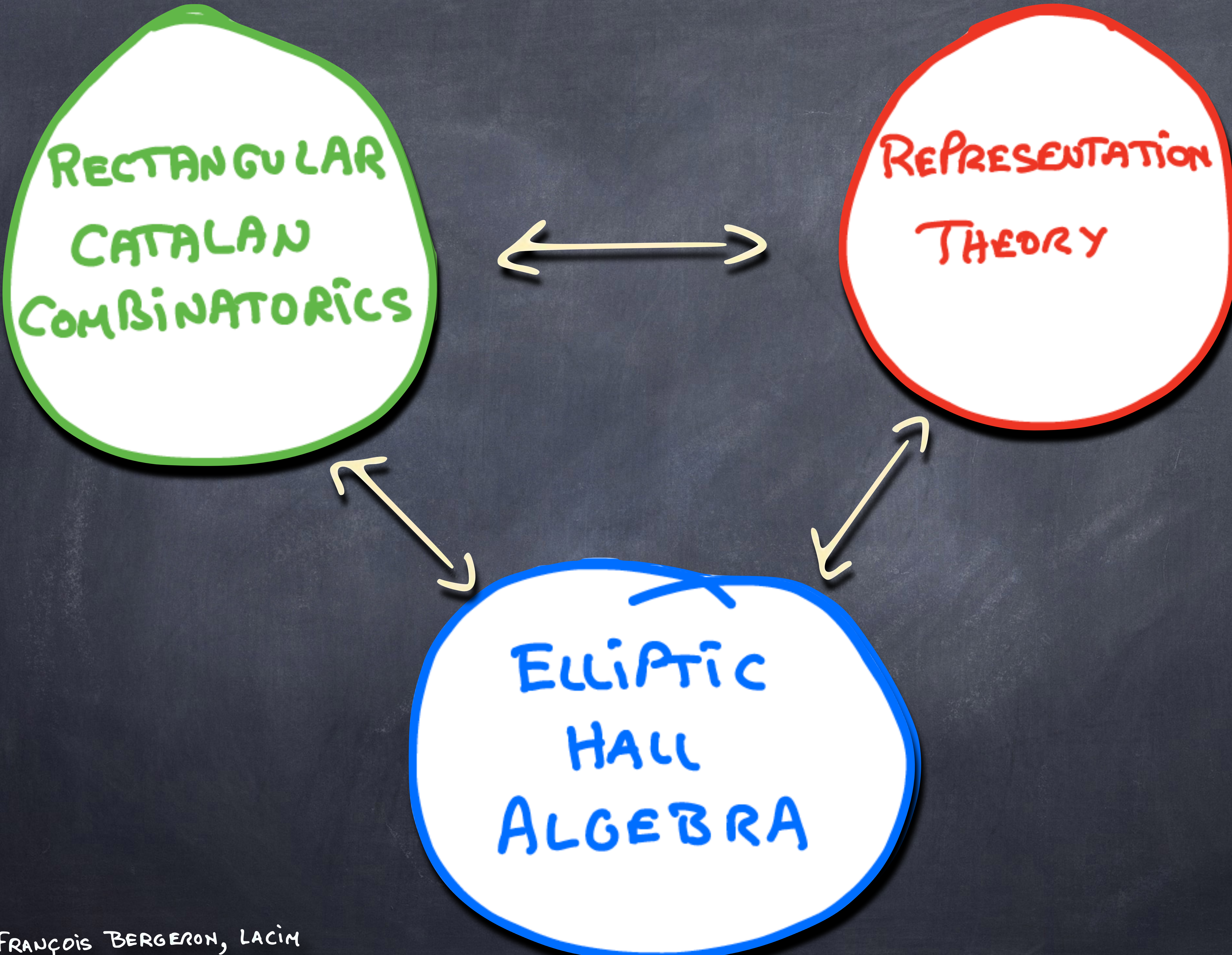

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Combinatoire de Catalan

Eugène Charles Catalan
(1814 – 1894)

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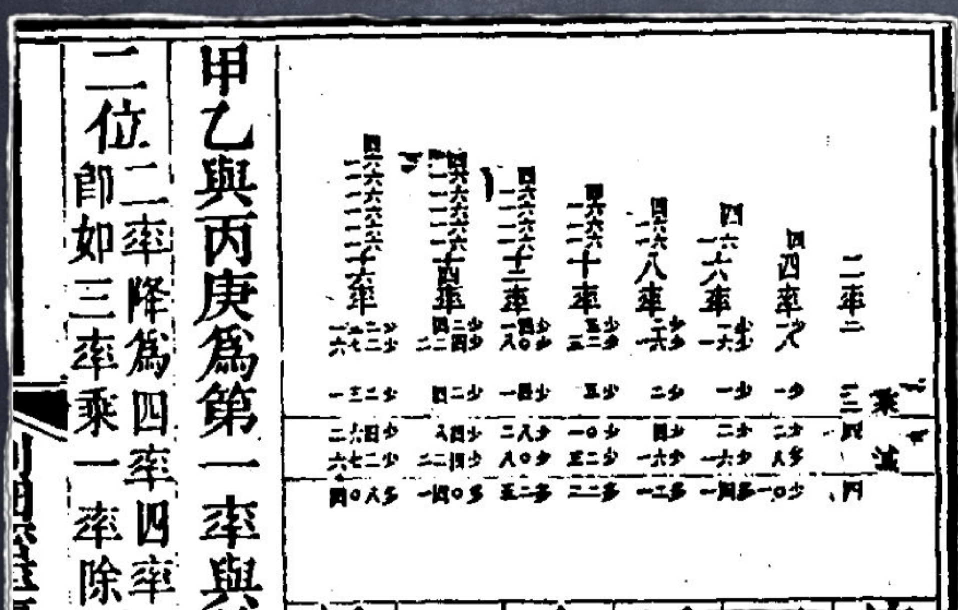
Nombres de Catalan

1 2 5 14 42 132 ...



Antu Ming (1692-1763) 1730

Johan Andreas von Segner (1704-1777) 1761

Eugène Charles Catalan (1814-1894) 1838

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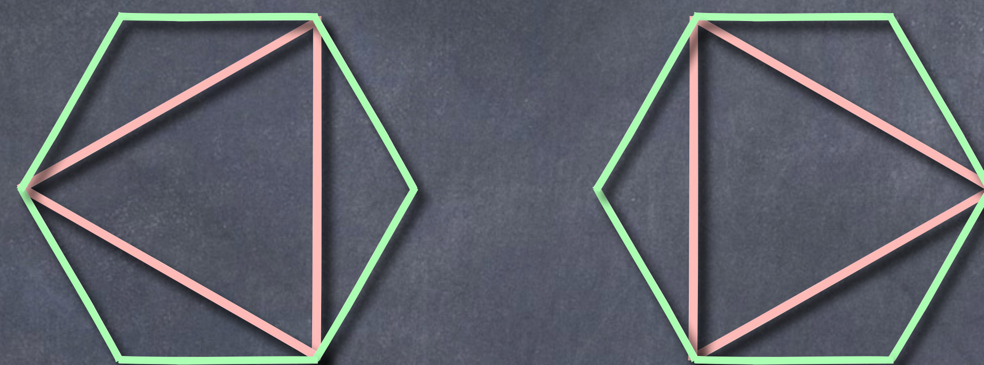
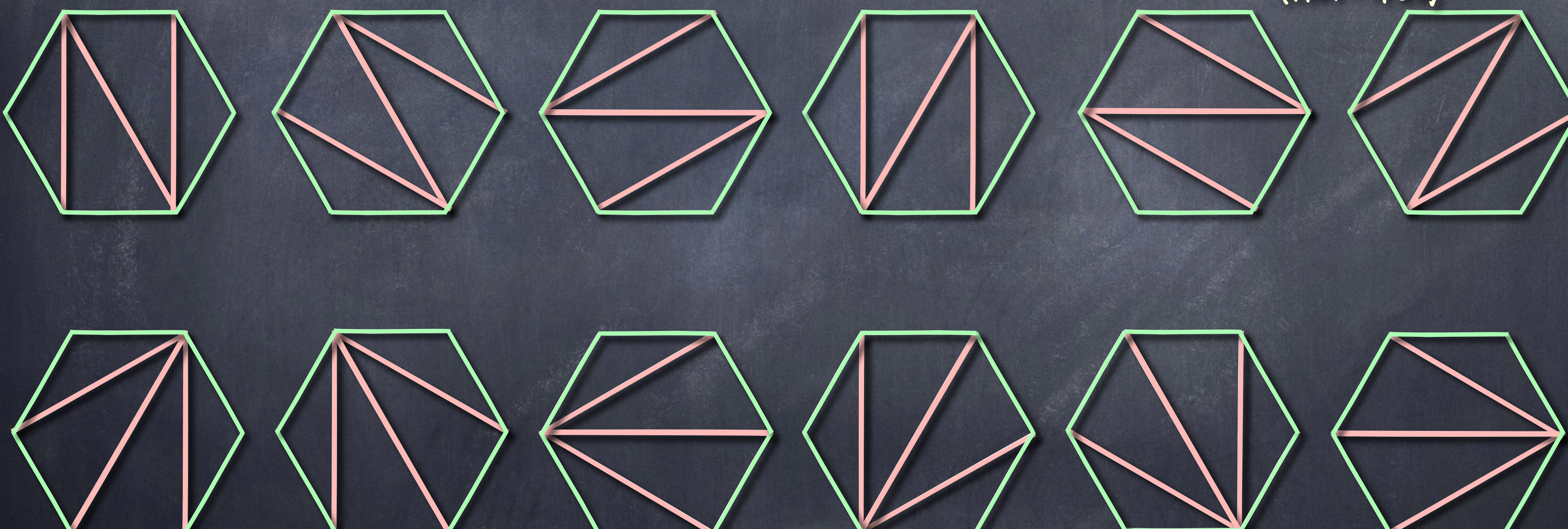
Triangulations

Leonhard Paul Euler
(1707 - 1783)





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$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$\mathcal{C}(z) := \sum_{n\geq 0} C_n z^n$$

$$\mathcal{C}(z) = 1 + z\,\mathcal{C}(z)^2$$

$$\frac{1-\sqrt{1-4z}}{2z} = 1 + 1\cdot z + 2\cdot z^2 + 5\cdot z^3 + 14\cdot z^4 + \ldots$$

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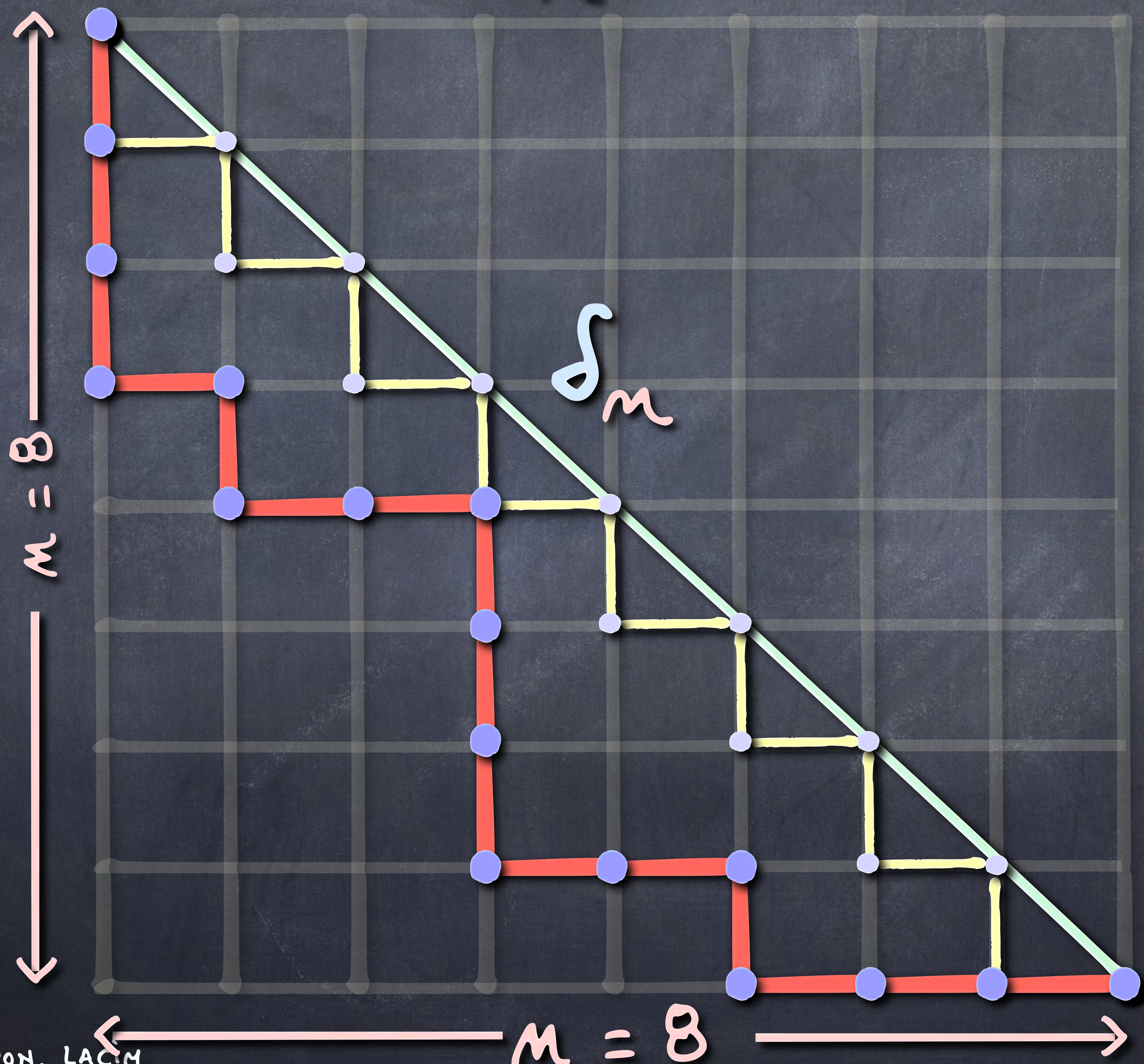
Partages contenus dans l'escalier δ_n

$\delta_n := (n-1, n-2, \ldots, 2, 1)$



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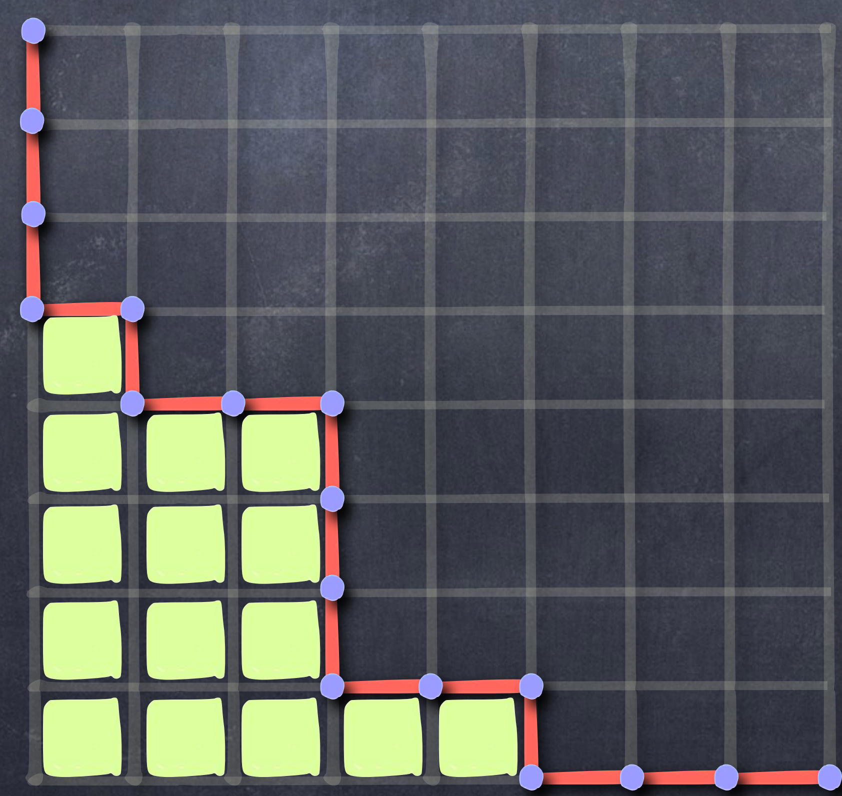
Enumération selon l'aire

$$C_n(q) := \sum_{\mu \subseteq \delta_n} q^{|\mu|}$$



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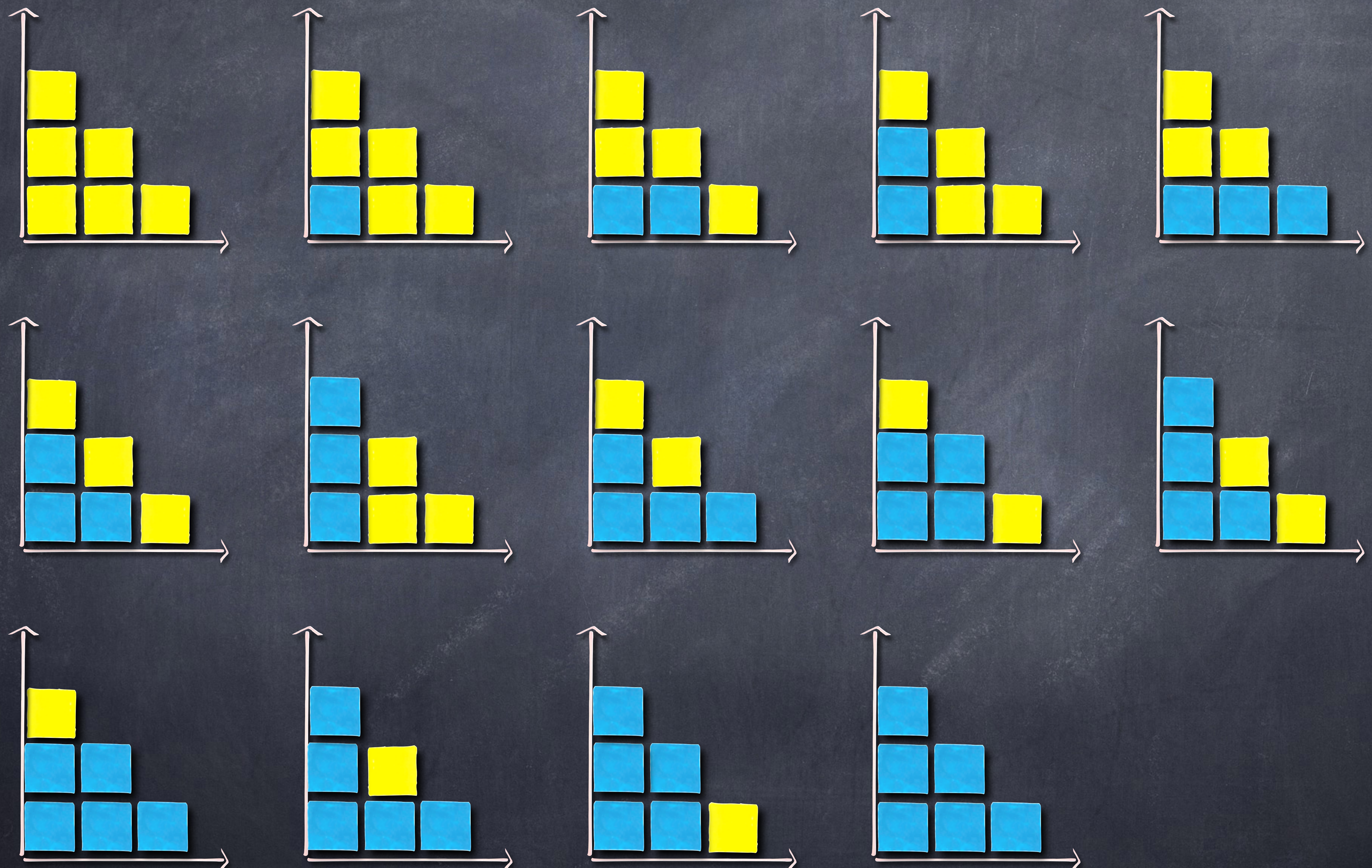
$$1+q+2q^2+3q^3+3q^4+3q^5+q^6$$



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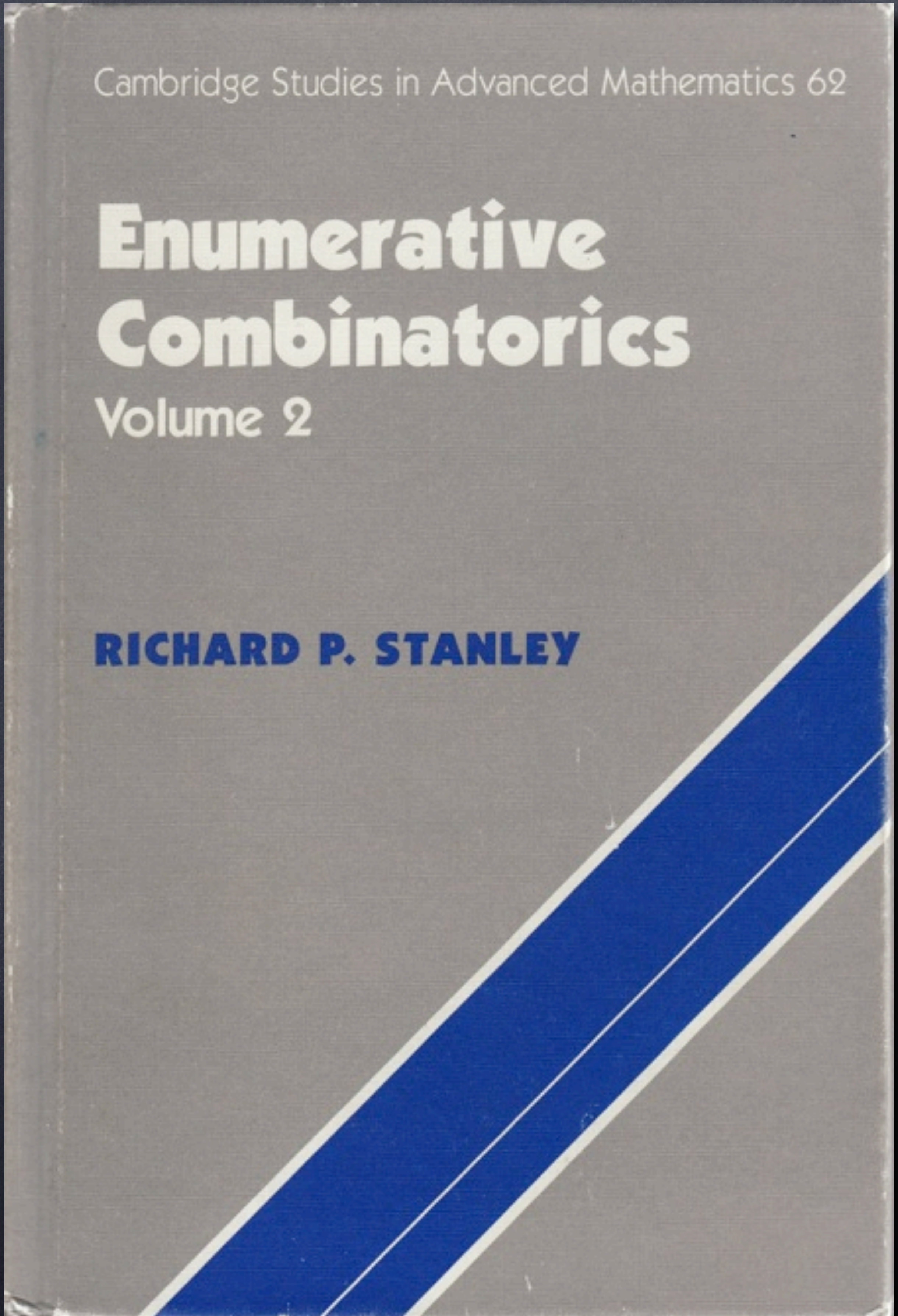

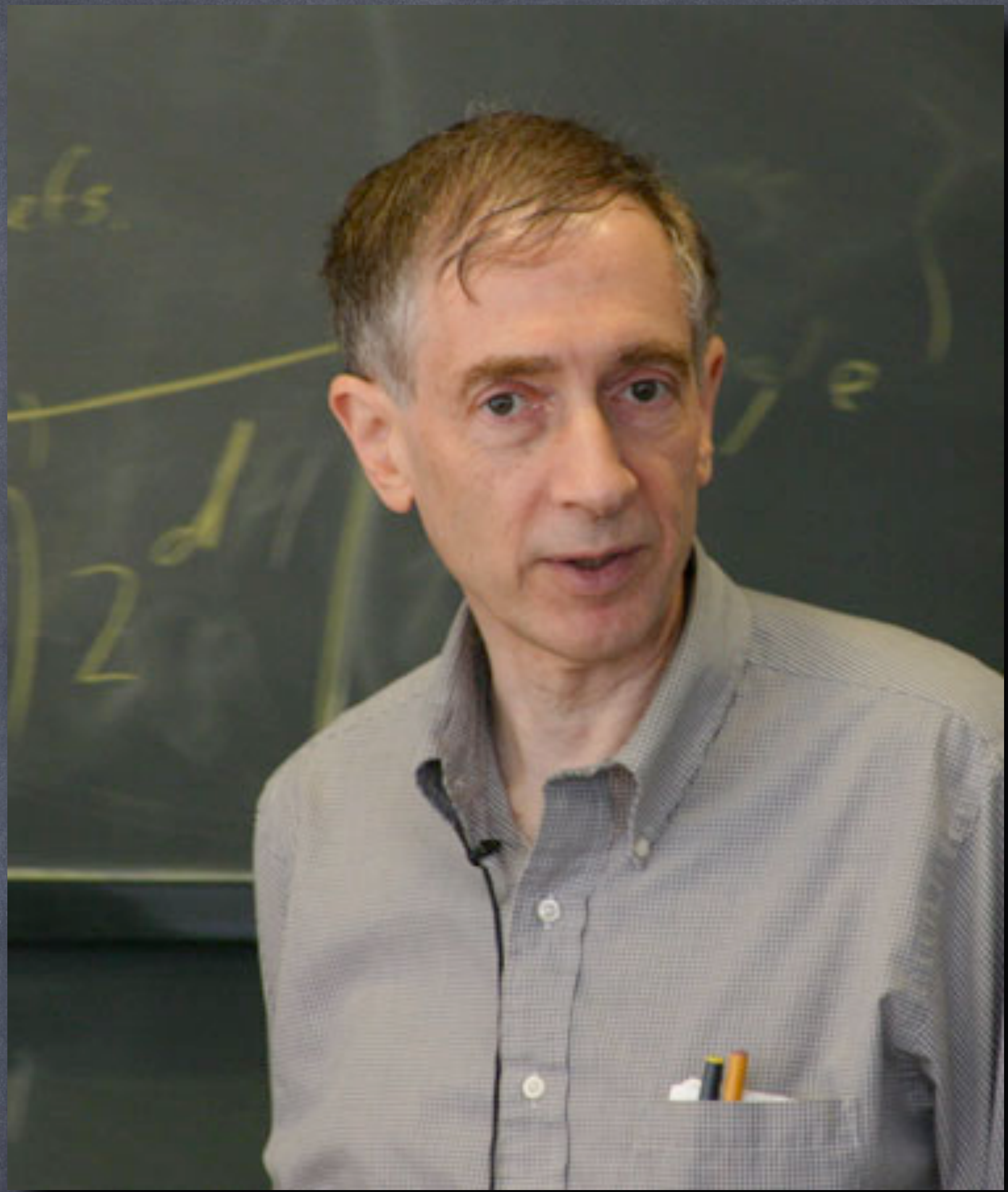

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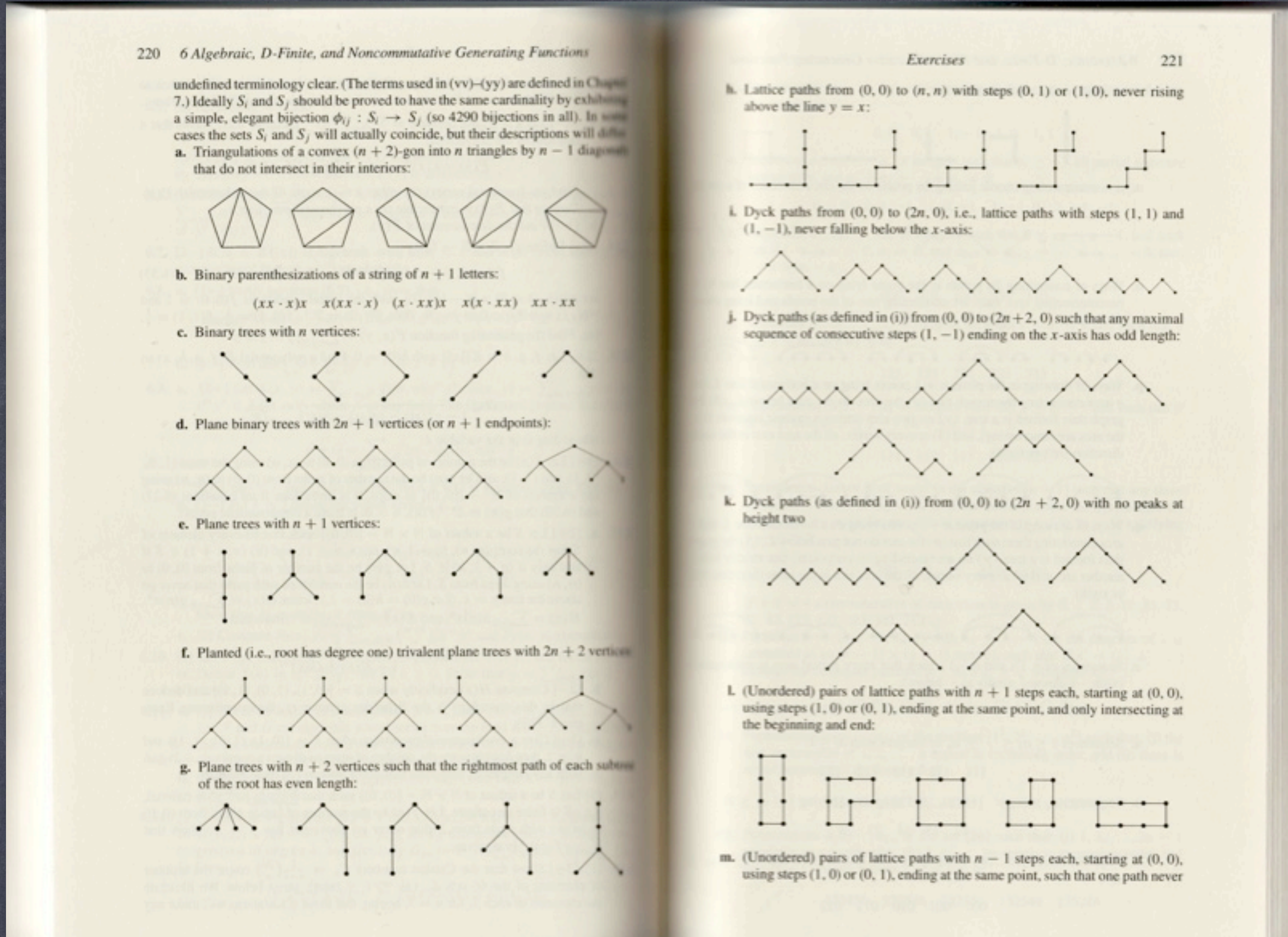

220 6 Algebraic, D-Finite, and Noncommutative Generating Functions

undefined terminology clear. (The terms used in (vv)–(yy) are defined in Chapter 7.) Ideally S_i and S_j should be proved to have the same cardinality by exhibiting a simple, elegant bijection $\phi_{ij} : S_i \to S_j$ (so 4290 bijections in all). In some cases the sets S_i and S_j will actually coincide, but their descriptions will differ.

a. Triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors:

b. Binary parenthesizations of a string of $n+1$ letters:

$(xx\cdot x)x \quad x(xx\cdot x) \quad (x\cdot xx)x \quad x(x\cdot xx) \quad xx\cdot xx$

c. Binary trees with n vertices:

d. Plane binary trees with $2n+1$ vertices (or $n+1$ endpoints):

e. Plane trees with $n+1$ vertices:

f. Planted (i.e., root has degree one) trivalent plane trees with $2n+2$ vertices:

g. Plane trees with $n+2$ vertices such that the rightmost path of each subtree of the root has even length:

Exercises 221

h. Lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ or $(1,0)$, never rising above the line $y=x$:

i. Dyck paths from $(0,0)$ to $(2n,0)$, i.e., lattice paths with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis:

j. Dyck paths (as defined in (i)) from $(0,0)$ to $(2n+2,0)$ such that any maximal sequence of consecutive steps $(1,-1)$ ending on the x-axis has odd length:

k. Dyck paths (as defined in (i)) from $(0,0)$ to $(2n+2,0)$ with no peaks at height two:

l. (Unordered) pairs of lattice paths with $n+1$ steps each, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, and only intersecting at the beginning and end:

m. (Unordered) pairs of lattice paths with $n-1$ steps each, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, such that one path never

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6.19. [1]–[3+] Show that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ count the number of elements of the 66 sets S_i, (a) $\le i \le$ (nnn), given below. We illustrate the elements of each S_i for $n = 3$, hoping that these illustrations will make any

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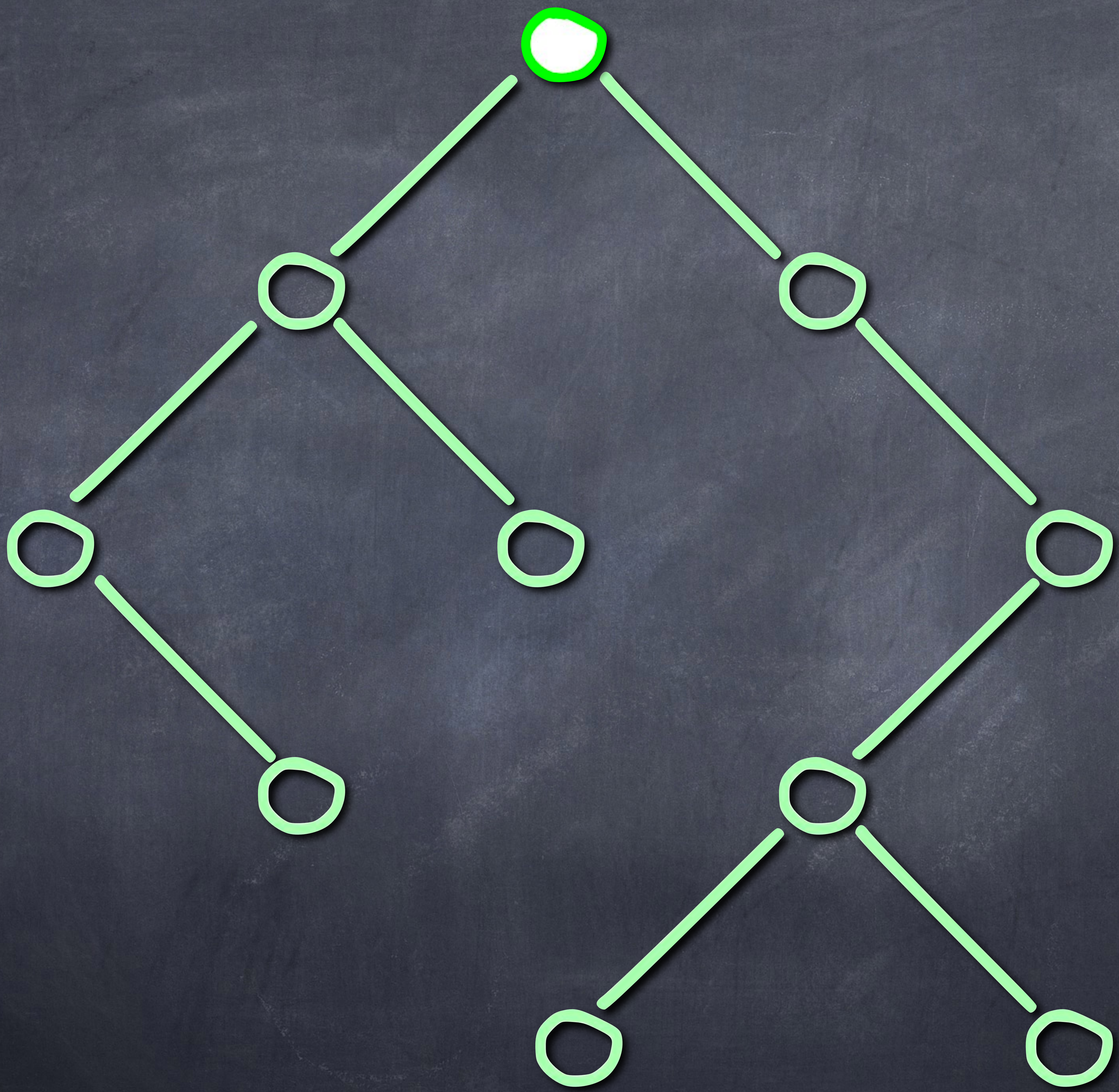

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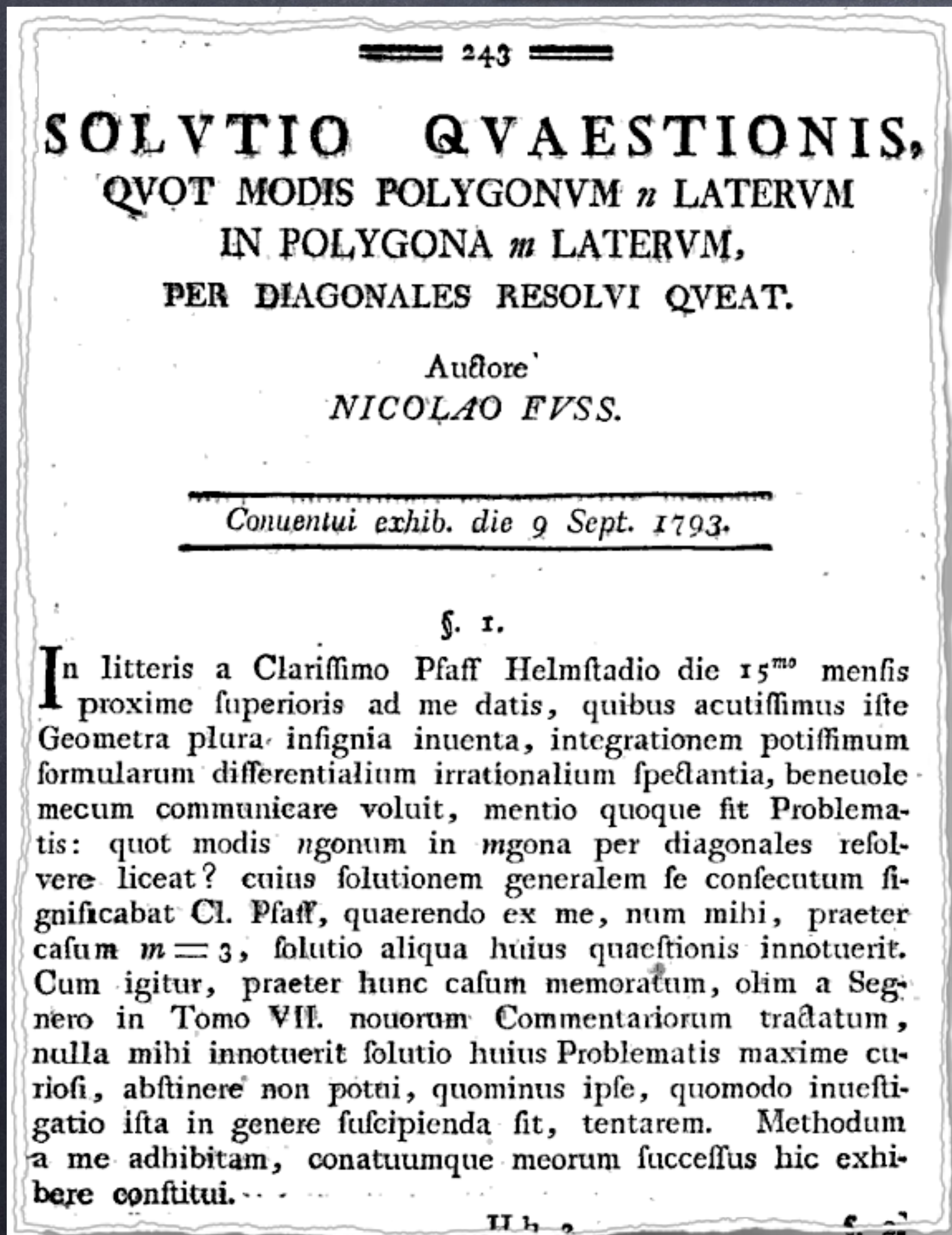
Nombres de Fuß-Catalan



SOLVTIO QVAESTIONIS,
QVOT MODIS POLYGONVM *n* LATERVM
IN POLYGONA *m* LATERVM,
PER DIAGONALES RESOLVI QVEAT.

Auctore
NICOLAO FVSS.

Conuentui exhib. die 9 Sept. 1793.

§. 1.

In litteris a Clariſſimo Pfaff Helmſtadio die 15^mo menſis proxime ſuperioris ad me datis, quibus acutiſſimus iſte Geometra plura inſignia inuenta, integrationem potiſſimum formularum differentialium irrationalium ſpectantia, beneuole mecum communicare voluit, mentio quoque fit Problematis: quot modis *n*gonum in *m*gona per diagonales reſoluere liceat? cuius ſolutionem generalem ſe conſecutum ſignificabat Cl. Pfaff, quaerendo ex me, num mihi, praeter caſum $m = 3$, ſolutio aliqua huius quaeſtionis innotuerit. Cum igitur, praeter hunc caſum memoratum, olim a Segnero in Tomo VII. nouorum Commentariorum tractatum, nulla mihi innotuerit ſolutio huius Problematis maxime curioſi, abſtinere non potui, quominus ipſe, quomodo inueſtigatio iſta in genere ſuſcipienda ſit, tentarem. Methodum a me adhibitam, conatuumque meorum ſucceſſus hic exhibere conſtitui.

1791

Nicolaus Fuss
(1755-1826)

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NOMBRES DE FUß-CATALAN

m

n

1	1	1	1	1	1	…
1	2	5	14	42	132	…
1	3	12	55	273	1428	…
1	4	22	140	969	7084	…

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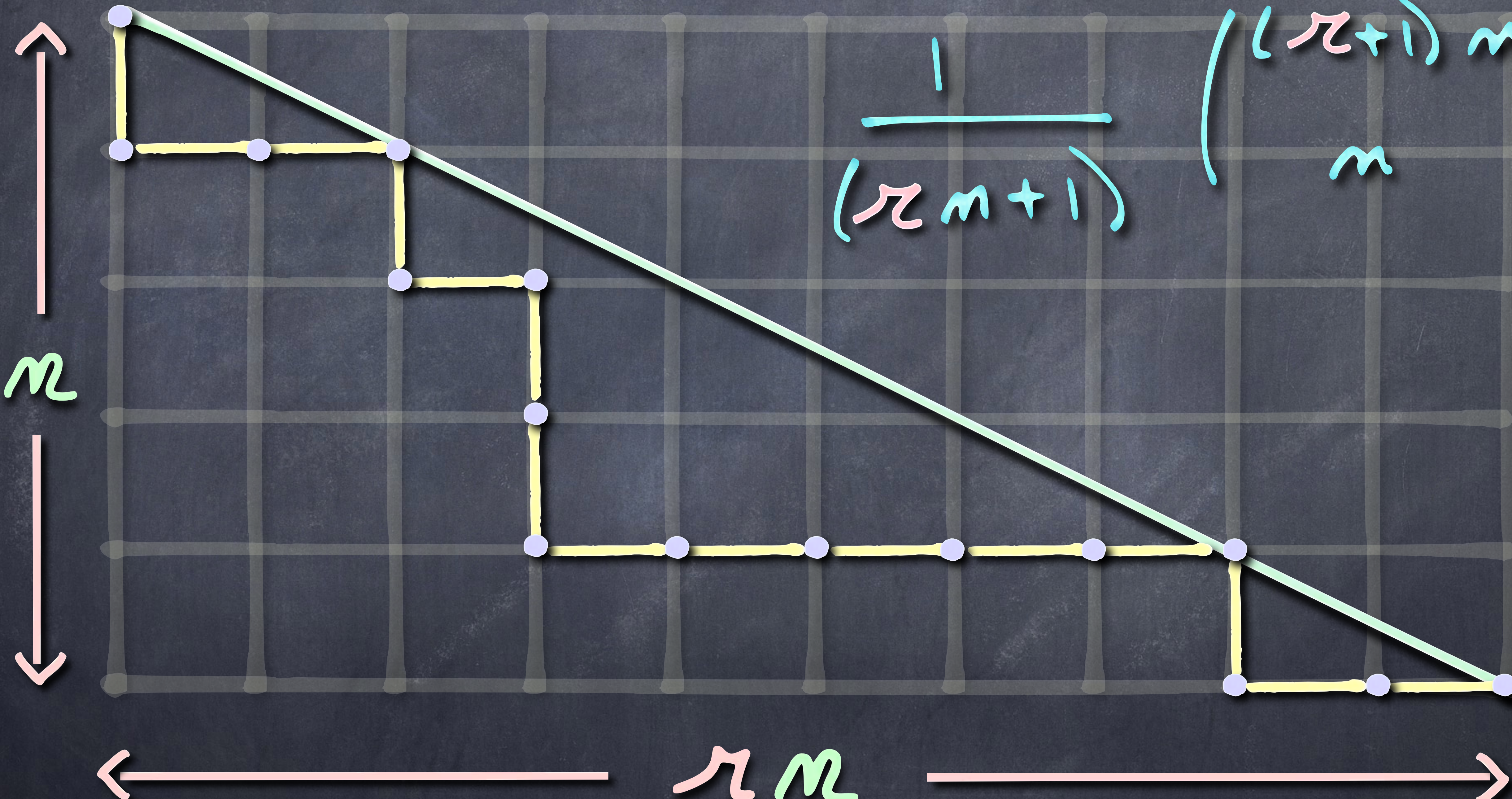
r-Chemins de Dyck



$$\frac{1}{(rn+1)}\binom{(r+1)n}{n}$$

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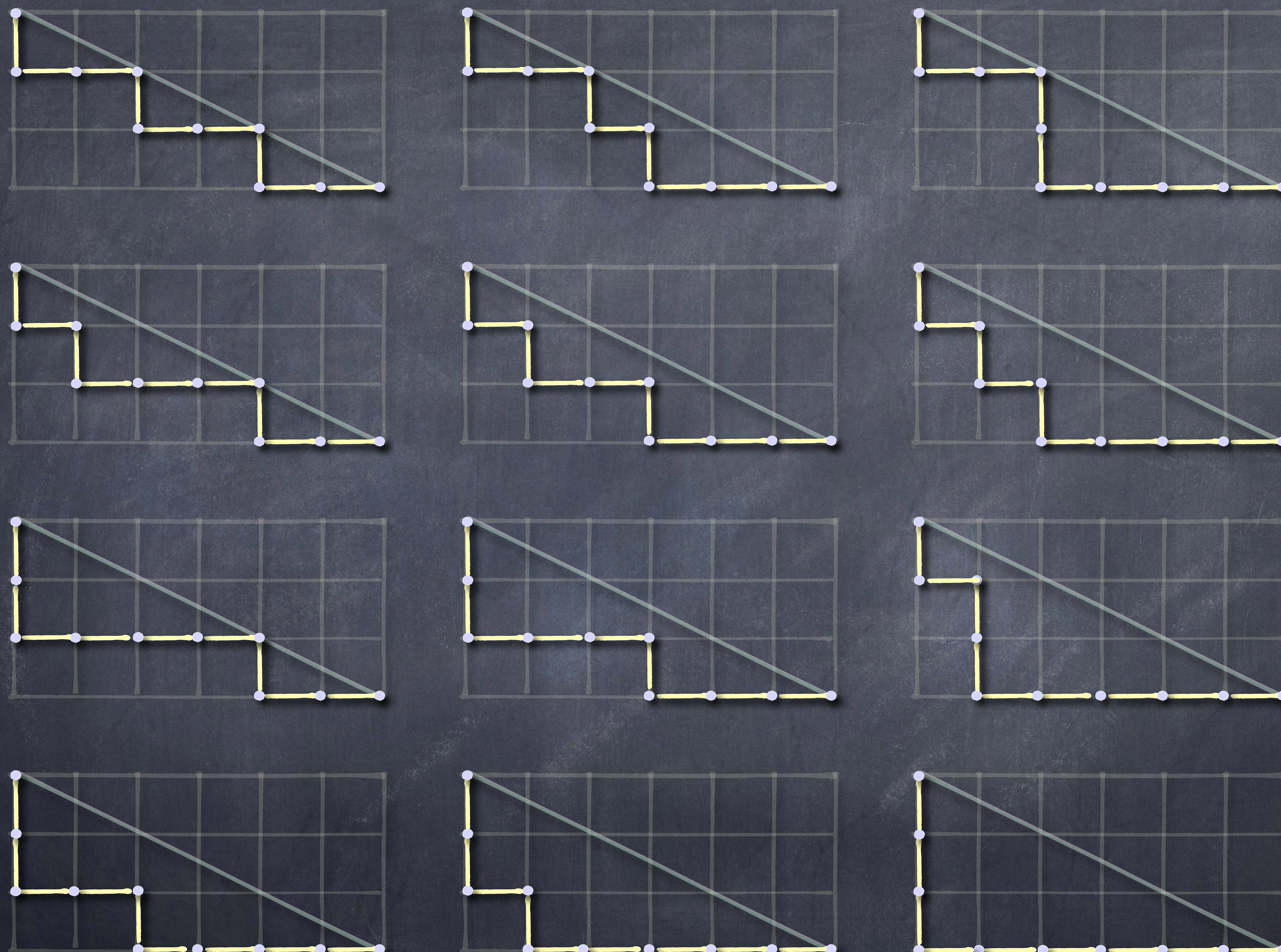

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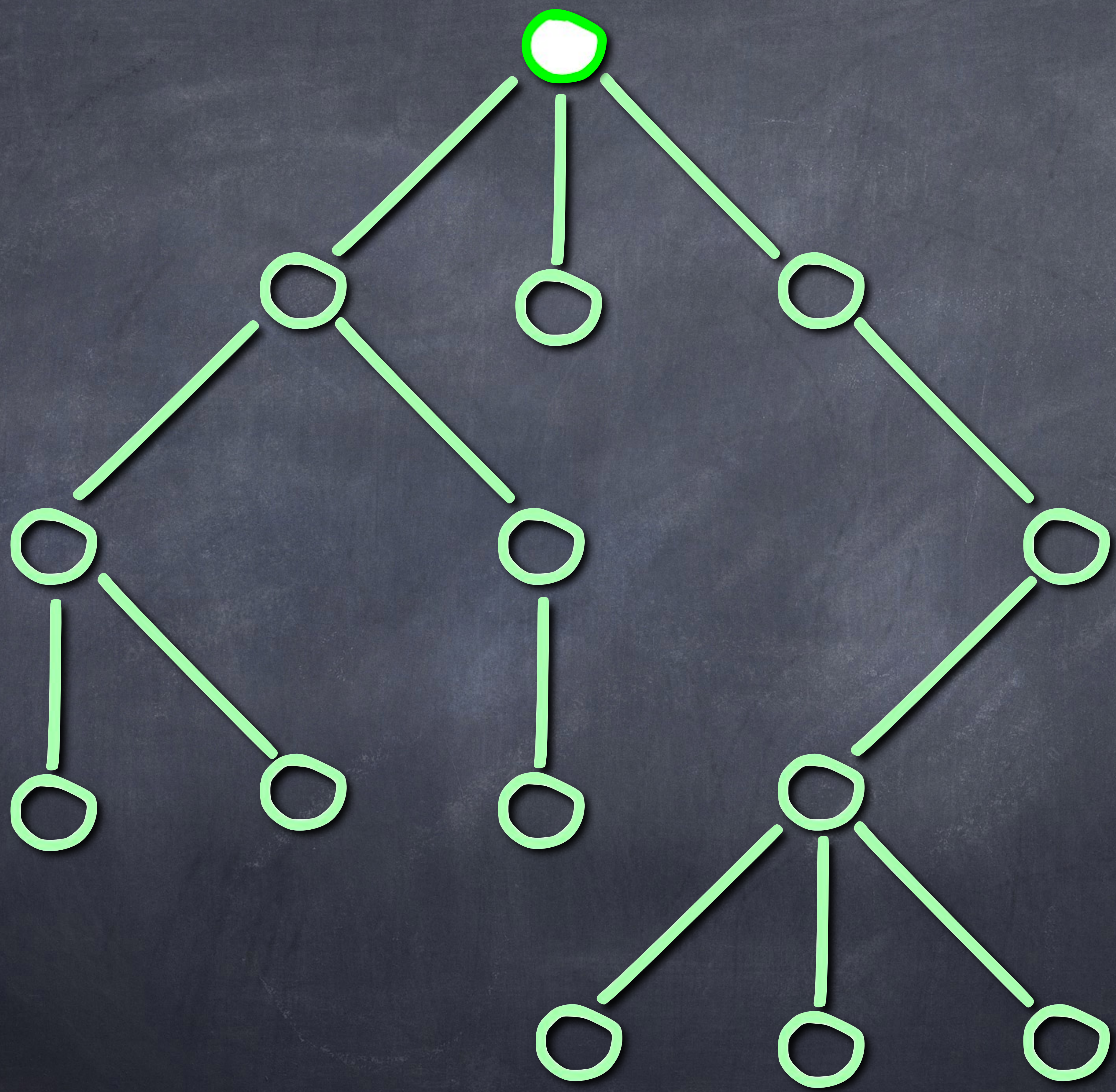

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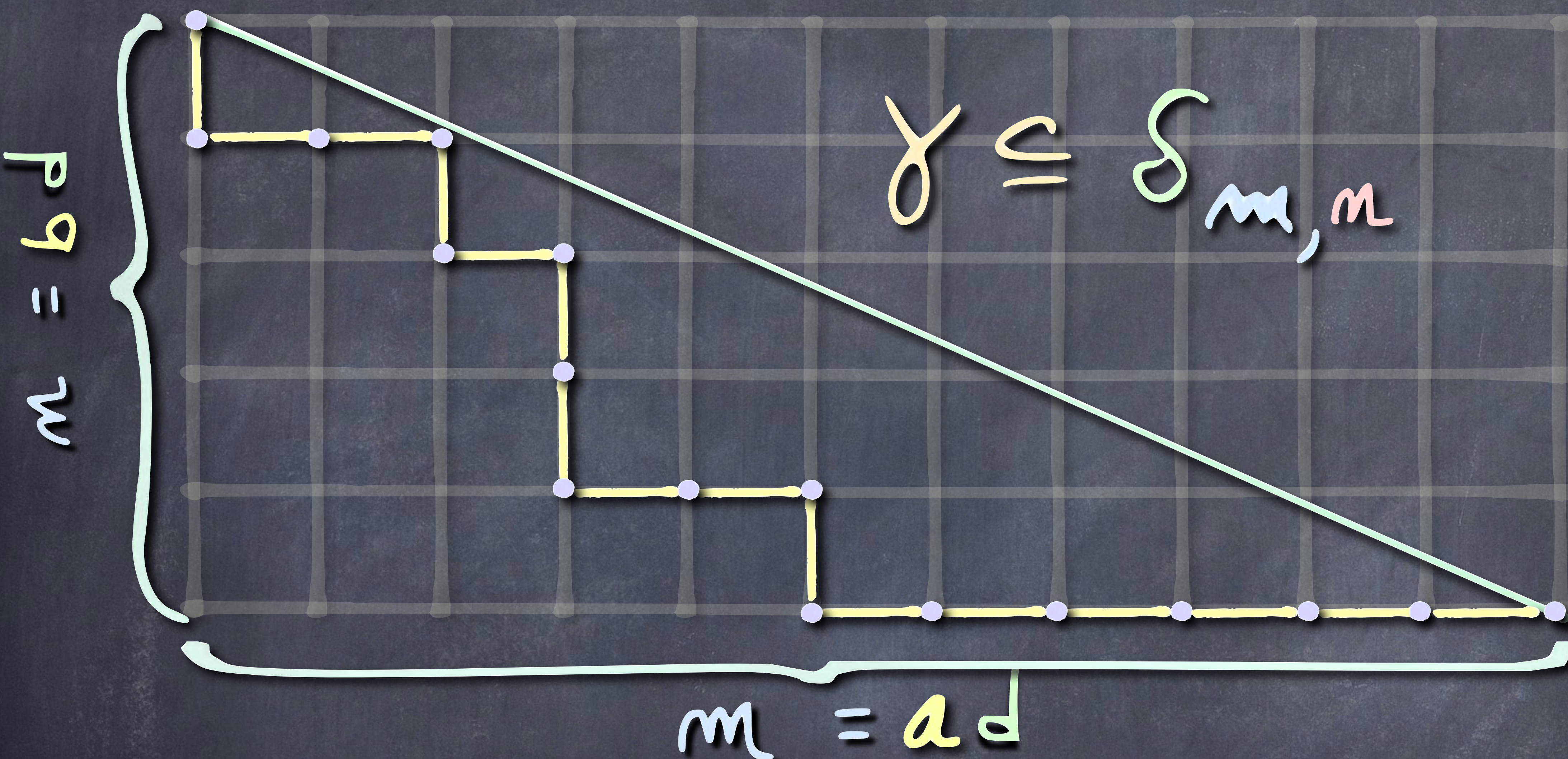
(m,n)-CHEMINS DE DYCK



$m = ad$

$n = bd$

$a \perp b$

COPREMIERS

(m,n)-ESCALIER $\qquad \delta_{m,n} := r_1 r_2 \cdots r_n$

$$r_k := \left\lfloor m(n-k)/n \right\rfloor$$

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Le début de la combinatoire de Catalan rectangulaire

DERIVATION OF A NEW FORMULA FOR THE NUMBER OF MINIMAL LATTICE PATHS FROM (0, 0) TO (km, kn) HAVING JUST t CONTACTS WITH THE LINE $my=nx$ AND HAVING NO POINTS ABOVE THIS LINE; AND A PROOF OF GROSSMAN'S FORMULA FOR THE NUMBER OF PATHS WHICH MAY TOUCH BUT DO NOT RISE ABOVE THIS LINE

BY M. T. L. BIZLEY, F.I.A., F.S.S., F.I.S.

WHITWORTH(1) deals in Chapter v of *Choice and Chance* with the problem of finding the number of minimal lattice paths from (0, 0) to (k, k) which do not cross the line $y=x$. By a lattice path is meant a path joining two points with integral coefficients by a line composed of horizontal and vertical steps of unit length. A minimal lattice path from (0, 0) to (x, y), say, is a lattice path where the total number of steps is $(x+y)$; in other words, all the steps are onwards. In what follows minimal lattice paths only will be considered, and the words 'minimal lattice' will be omitted.

Although Whitworth deals only with the case where the boundary line (i.e. the line which the path must not cross) is $y=x$, the more general case of a boundary $\alpha y=x$ has been solved provided α is a positive integer(2), (3). The number of paths from (0, 0) to (αl, l) which may touch but never rise above $\alpha y=x$ is $\frac{1}{l\alpha+1}\binom{l\alpha+l}{l}$.

Grossman(4) announced without proof in 1950 a formula for the number of paths from (0, 0) to (km, kn) which may touch but never rise above the line $my=nx$, where k is a positive integer and m and n are coprime positive integers; thus (km, kn) is any point having positive integral coefficients. Grossman's formula is

$$\Sigma F_1^{k_1} F_2^{k_2} \ldots / k_1!\, k_2! \ldots,$$

where

$$F_j = \frac{1}{j(m+n)}\binom{jm+jn}{jm},$$

the sum extending over all positive integral k_i such that $k_i \geqslant 0$ and $\Sigma i k_i = k$. If $k=1$ this takes the simple form $\frac{1}{m+n}\binom{m+n}{m}$.

Bizley 1954

Journal of the Institute of Actuaries

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Formule de Bizley

$$\mathrm{Cat}_{(ad,bd)} = \sum_{\mu \vdash d} \frac{1}{z_\mu} \prod_{k \in \mu} \frac{1}{a+b} \binom{ak+bk}{ak}$$

Parts

μ Partition of d

$d = \mu_1 + \mu_2 + \cdots + \mu_\ell \qquad \mu_i \geq \mu_{i+1}$

$z_\mu := 1^{r_1} r_1!\, 2^{r_2} r_2! \cdots d^{r_d} r_d!$

$r_i :=$ # Parts of μ of size i

$m = ad$

$n = bd$

$a \perp b$

Copremiers

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Fonctions « Parking »

Linear hashing

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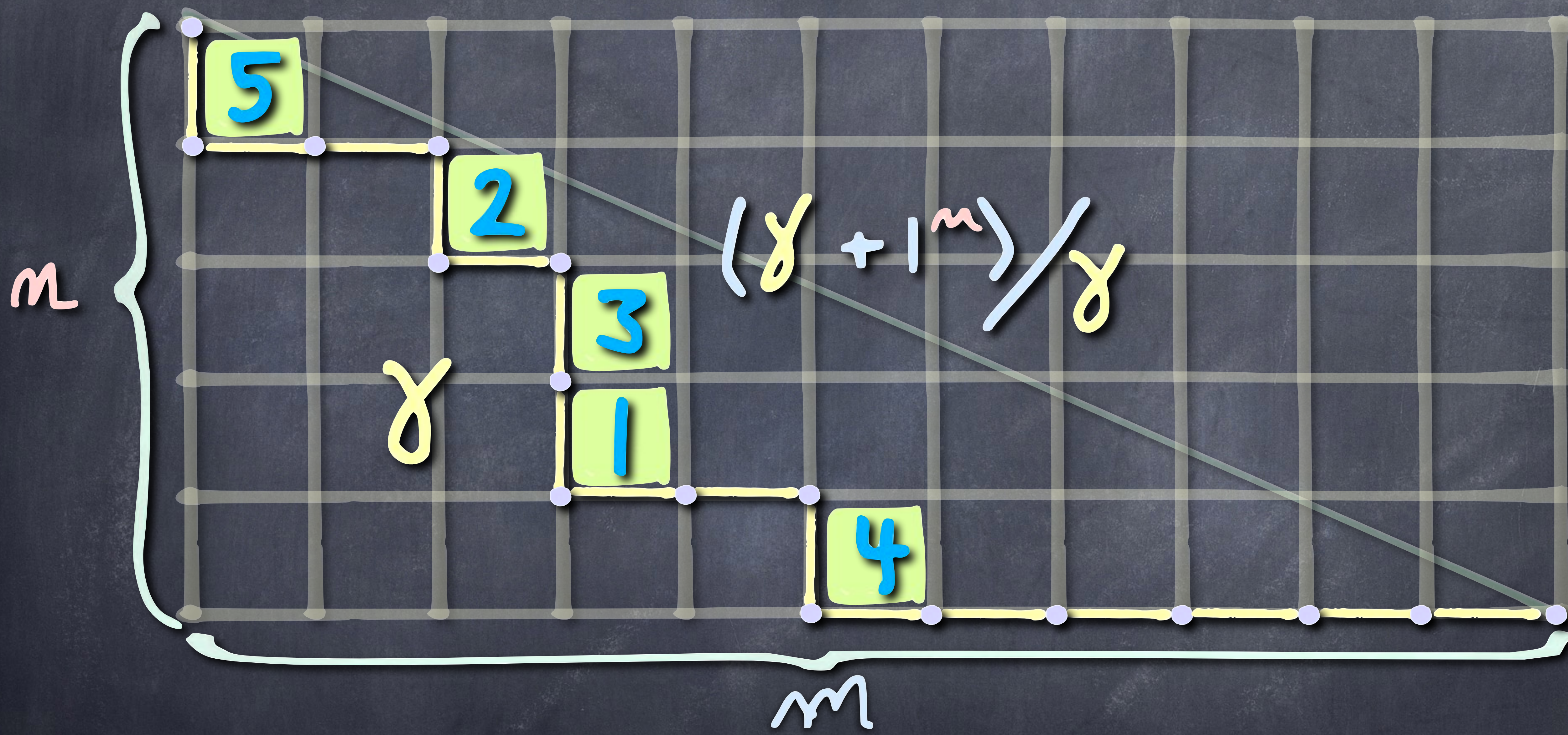
(m,n)-Fonctions « Parking »



Tableaux standards de forme gauche : $(\gamma+1^m)/\gamma$

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(m,n)-Fonctions « Parking »

$$\#\mathrm{Park}_{n,n} = (n+1)^{(n-1)}$$

PGCD$(m,n) = 1$

$$\#\mathrm{Park}_{m,n} = m^{(n-1)}$$

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À LA BIZLEY-GROSSMAN

$$\sum_{d=0}^{\infty} \#\mathrm{PARK}_{da,db} \frac{x^{db}}{(db)!} = e^{\sum_{j\geq 1} (ja)^{jb-1} \frac{x^{jb}}{(jb)!}}$$

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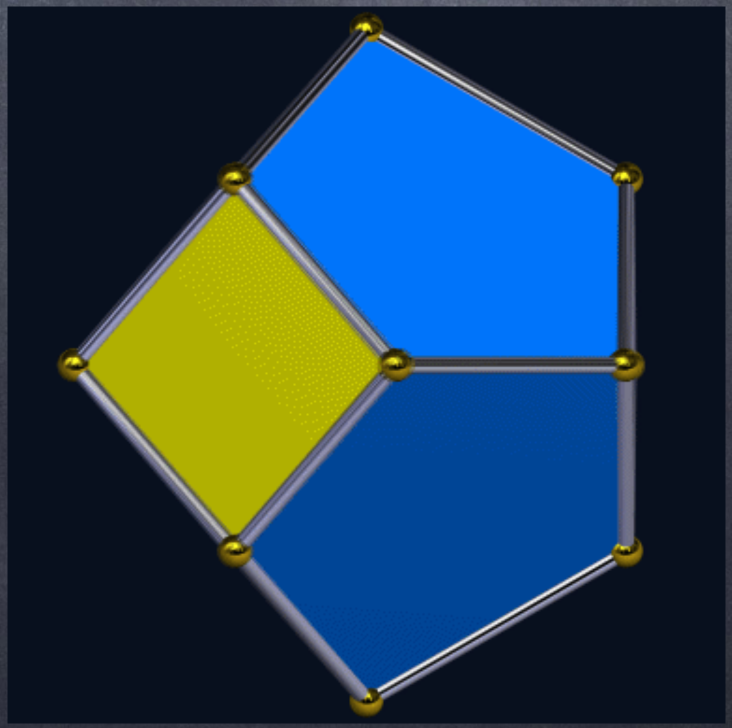
Treillis de Tamari



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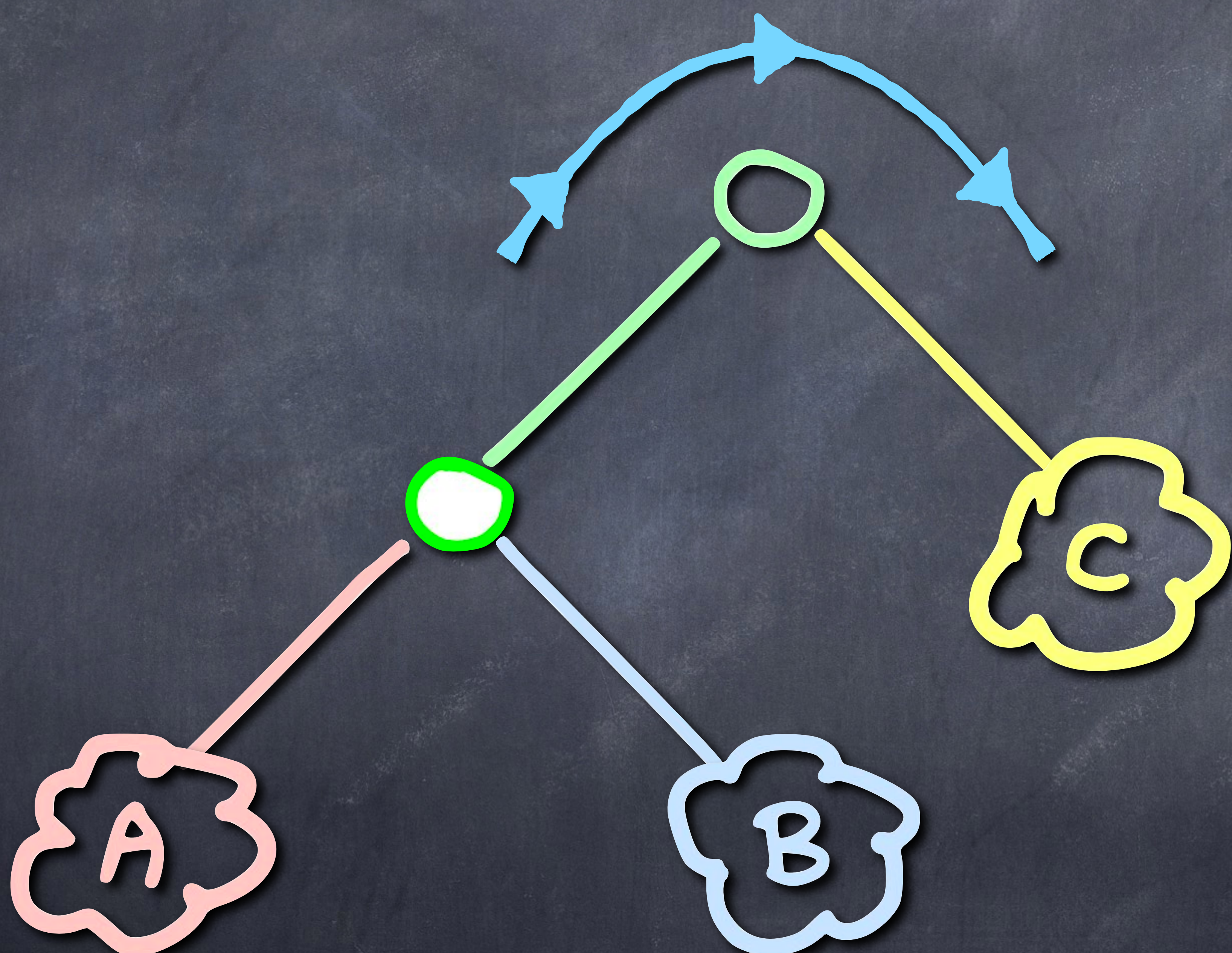
Rotations à droite



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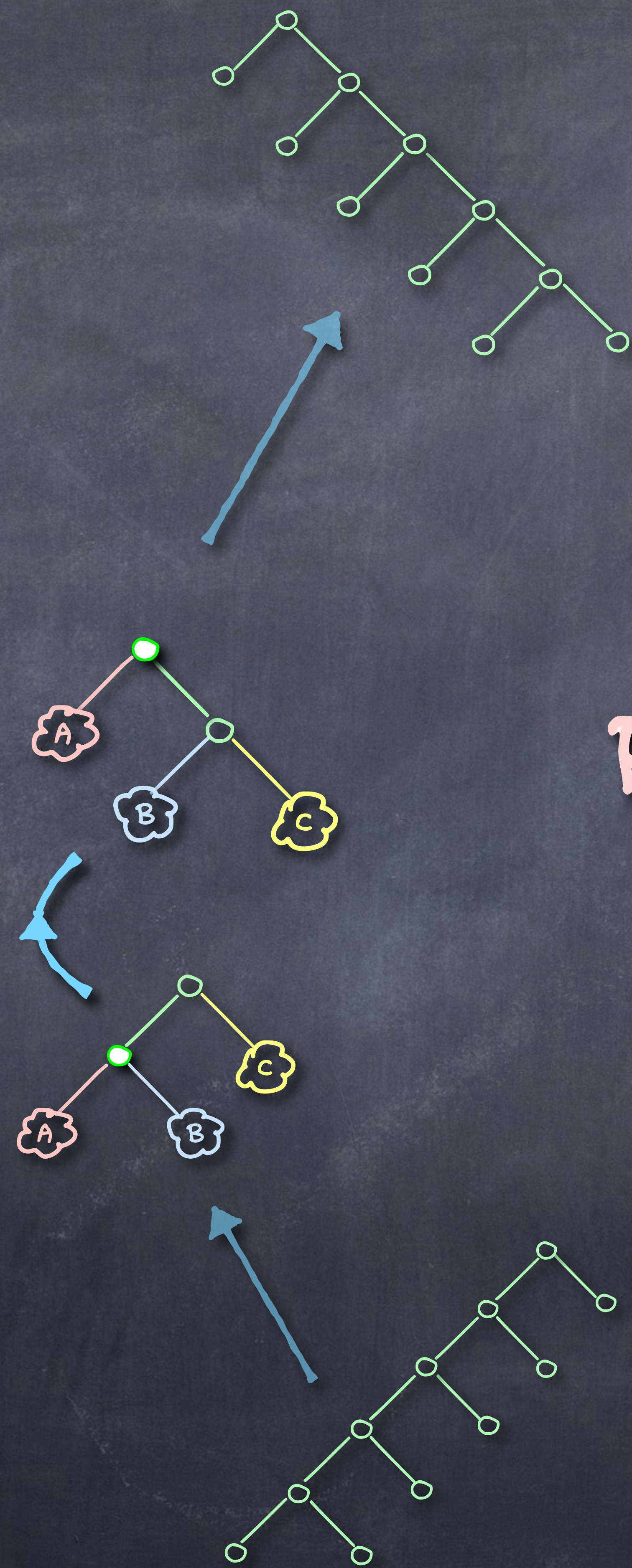
Rotations à droite



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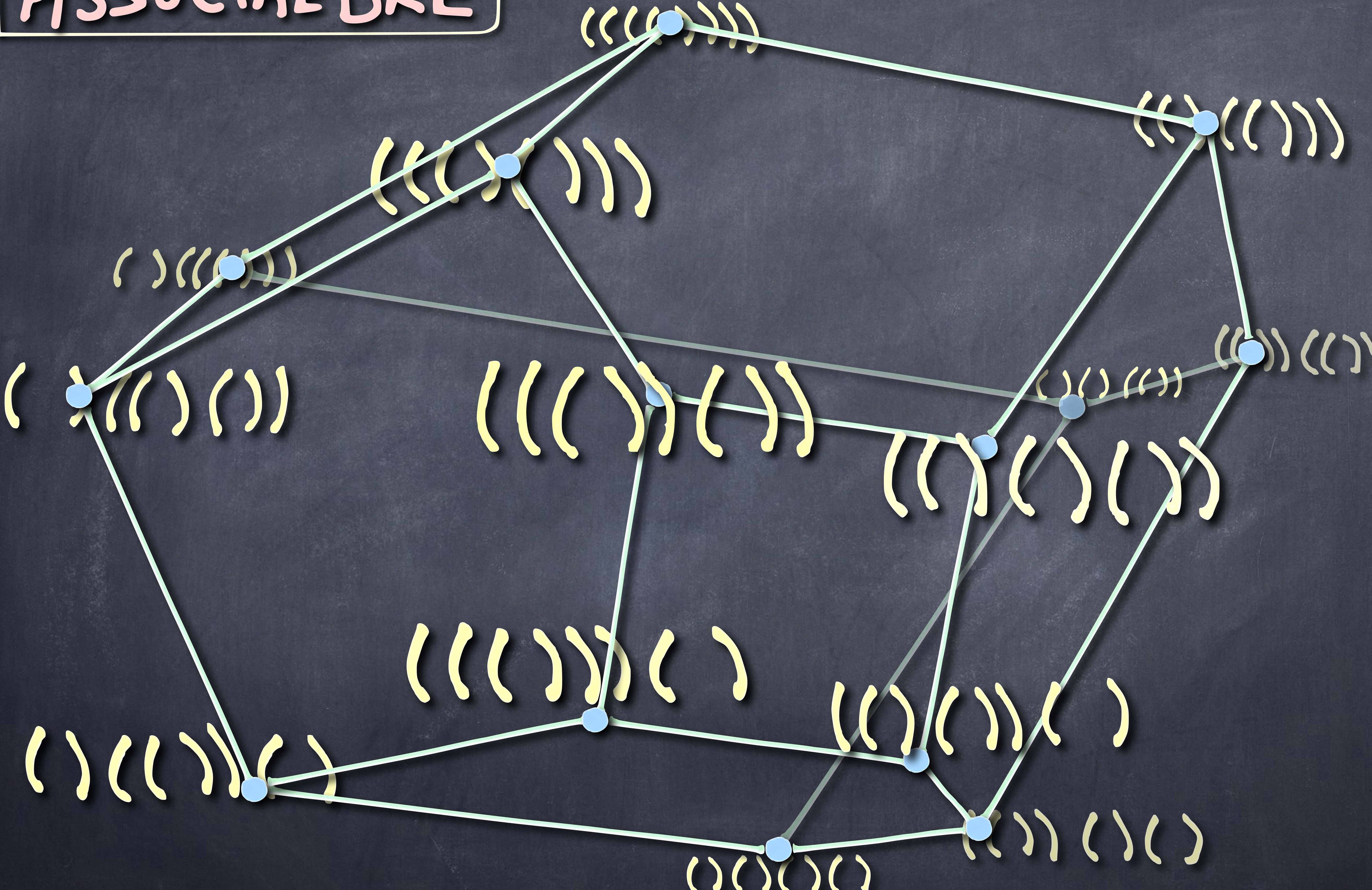
Associaèdre



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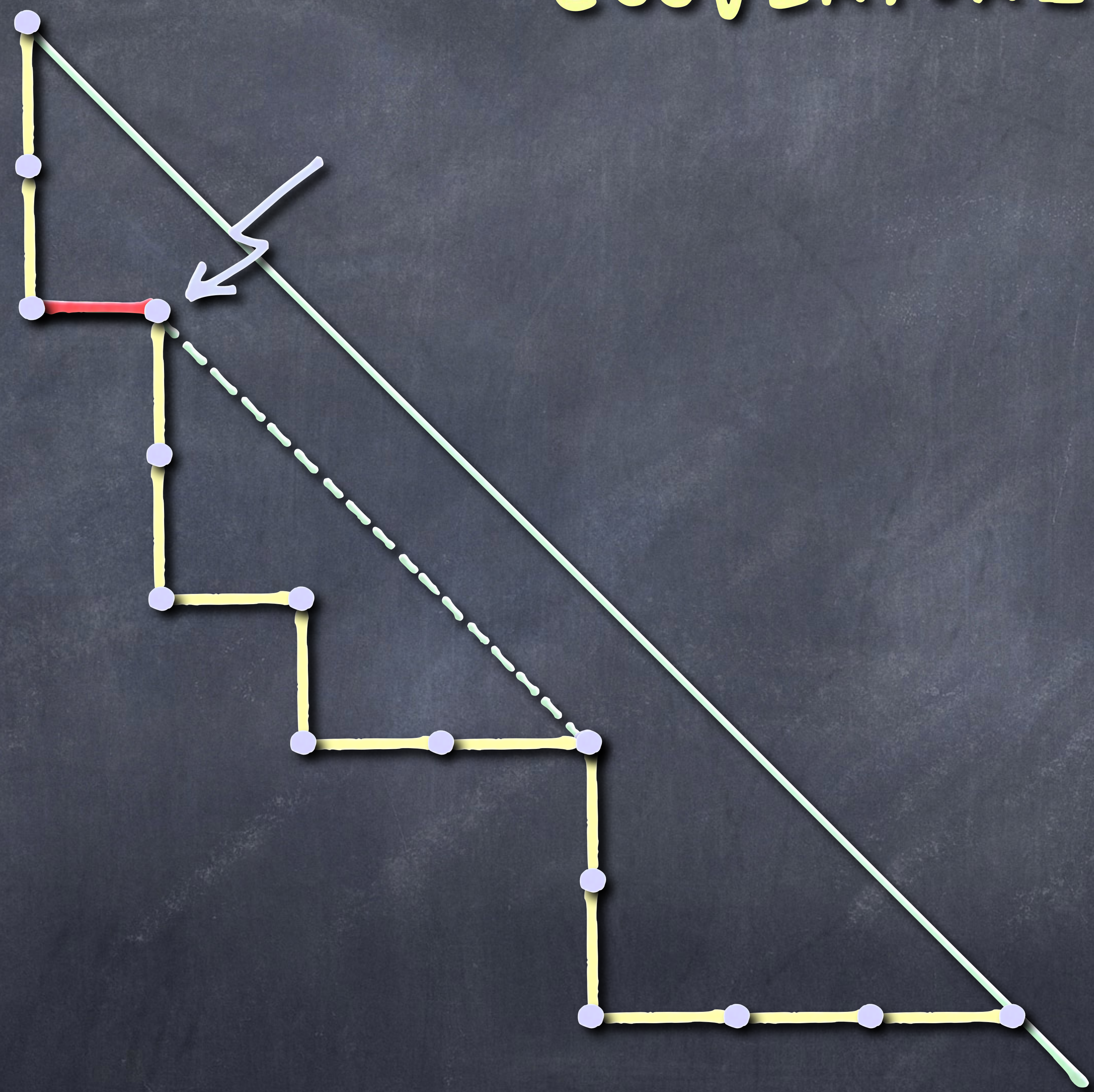
TAMARI RELATION DE COUVERTURE



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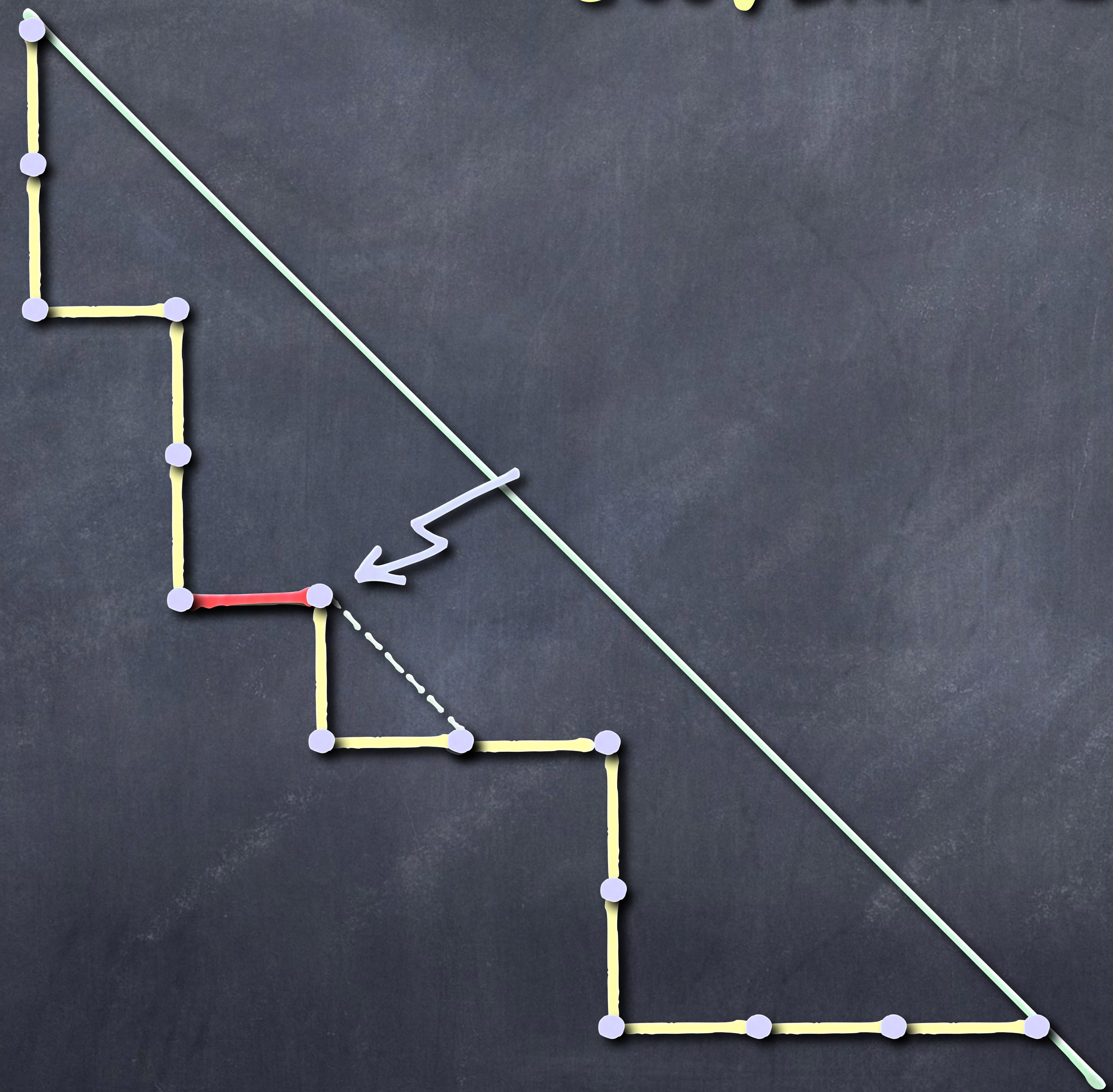
TAMARI RELATION DE COUVERTURE



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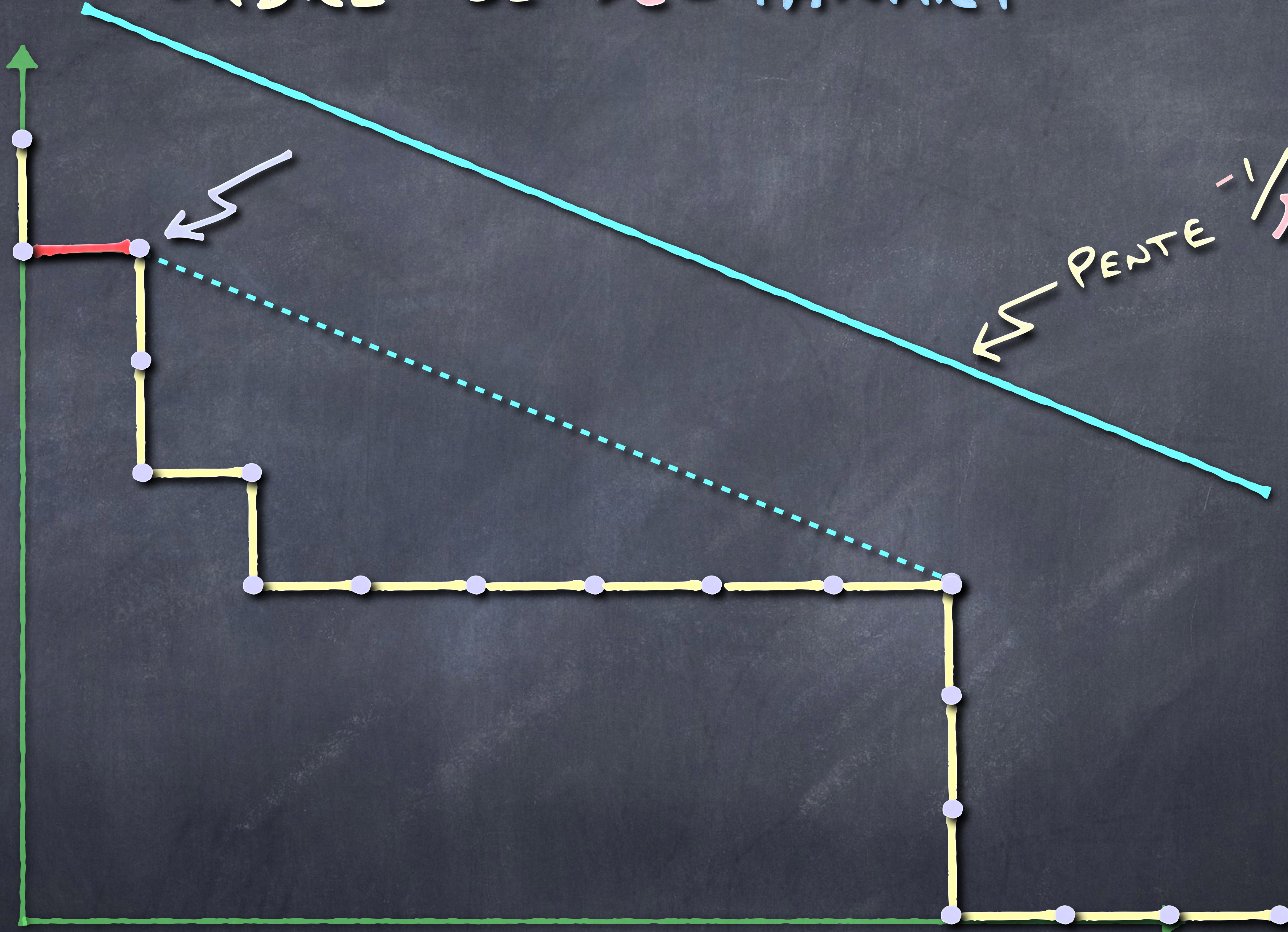
ORDRE DE r-TAMARI



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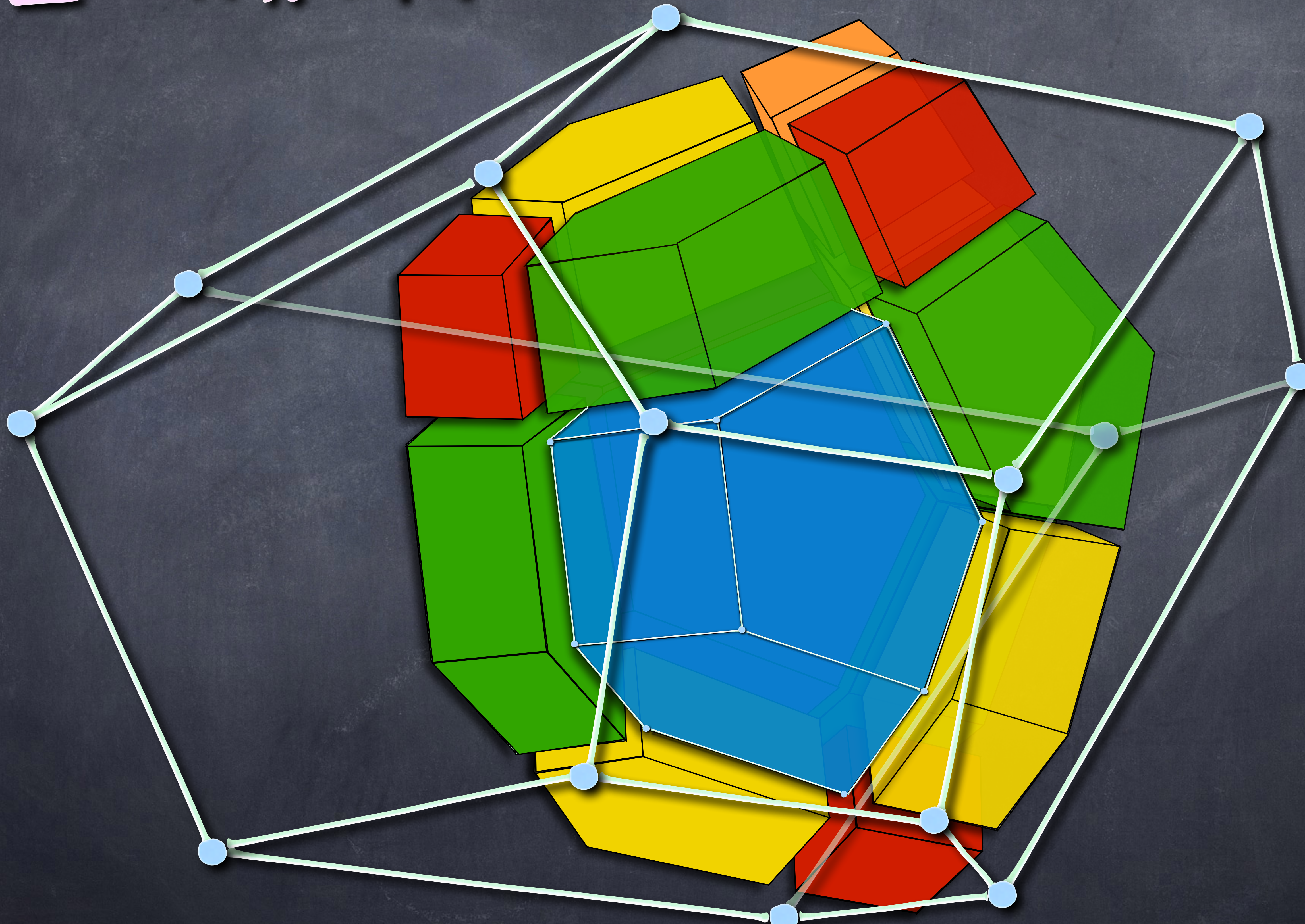
2-TAMARÎ



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Polynômes Harmoniques Diagonaux

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Polynômes harmoniques diagonaux

ℓ jeux de variables

$$\mathcal{D}_n := \{\, g(x) \mid f(\partial X)\, g(x) = 0,\ f(0)=0 \,\}$$

$$\downarrow$$

Pour tout $f(\partial X)$ diagonalement symétrique

$$f(\partial X) = \sum_A a_A\, \partial x_1^{a_1} \cdots \partial x_n^{a_n}\, \partial y_1^{b_1} \cdots \partial y_n^{b_n} \cdots \partial z_1^{c_1} \cdots \partial z_n^{c_n}$$

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ALTERNANTS

$$\mathcal{A}_n := \left\{ g(x) \in \mathcal{D}_n \;\middle|\; g(x\sigma) = (-1)^{\mathrm{inv}(\sigma)}\, g(x), \text{ pour tout } \sigma \in S_n \right\}$$

CAS $\ell = 1$

$$\mathcal{D}_3 = \mathbb{Q}[\partial x^\alpha \Delta_3 \mid \alpha \in \mathbb{N}^3]$$

$$\Delta_3 = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

$$\mathcal{D}_3 = \mathbb{Q}\{\Delta_3, \partial x_1 \Delta_3, \partial x_2 \Delta_3, \partial x_1^2 \Delta_3, \partial x_1 \partial x_2 \Delta_3, 1\}$$

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CAS $\ell = 1$

$$\dim(\mathcal{D}_n) = n!$$

$$\dim(\mathcal{A}_n) = 1$$

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CAS $\ell = 2$

$$\dim(\mathcal{D}_n) = (n+1)^{n-1}$$

$$\dim(A_n) = \frac{1}{n+1}\binom{2n}{n}$$

CAS $\ell = 3$

$$\dim(\mathcal{D}_n) = 2^n (n+1)^{n-2}$$

$$\dim(A_n) = \frac{2}{n(n+1)} \binom{4n+1}{n-1}$$

QUESTION ALGÉBRIQUE OUVERTE

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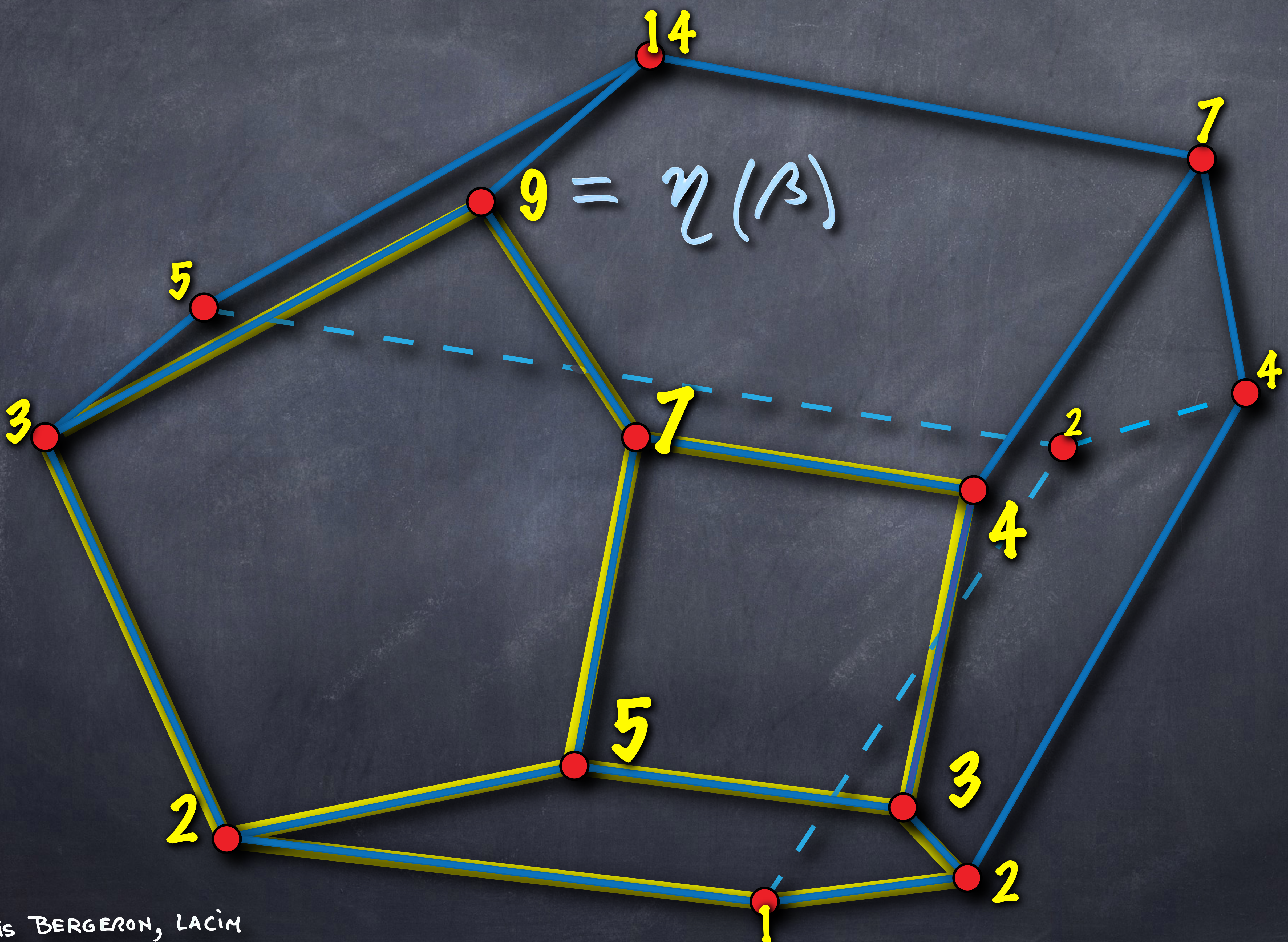
$\eta(\beta)$: NOMBRE D'ÉLÉMENTS $\leq \beta$



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Dimensions

$1^\ell := \underbrace{11\cdots 1}_{\ell\text{-copies}}$

$$\mathcal{D}_1(1^\ell) = 1$$

$$\mathcal{D}_2(1^\ell) = 1 + \binom{\ell}{1}$$

$$\mathcal{D}_3(1^\ell) = 1 + 2\binom{\ell}{1} + \binom{\ell}{1}^2 + \binom{\ell+1}{2} + \binom{\ell+2}{3}$$

$$\mathcal{D}_4(1^\ell) = 1 + 3\binom{\ell}{1} + 3\binom{\ell}{1}^2 + 2\binom{\ell+1}{2} + \binom{\ell}{1}^3$$
$$+3\binom{\ell}{1}\binom{\ell+1}{2} + 2\binom{\ell+2}{3} + 4\binom{\ell}{1}\binom{\ell+2}{3}$$
$$+\binom{\ell+3}{4} + \binom{\ell}{1}\binom{\ell+3}{4} + 2\binom{\ell+4}{5} + \binom{\ell+5}{6}$$

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Dimensions Alternants

$$A_1(1^\ell) = 1$$

$$A_2(1^\ell) = 1 + \binom{\ell-1}{1}$$

$$A_3(1^\ell) = 1 + 2\binom{\ell-1}{1} + \binom{\ell-1}{1}^2 + \binom{\ell+1}{3}$$

$$A_4(1^\ell) = 1 + 3\binom{\ell-1}{1} + 3\binom{\ell-1}{1}^2 + \binom{\ell-1}{1}^3 + 2\binom{\ell+1}{3} + 2\binom{\ell-1}{1}\binom{\ell+1}{3} + \binom{\ell-1}{1}\binom{\ell+2}{4} + \binom{\ell+4}{6}$$

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- Versions rectangulaires
- Théorie des noeuds
- Combinatoire algébrique
- Physique théorique
- Géométrie algébrique
- Théorie des fonctions symétriques

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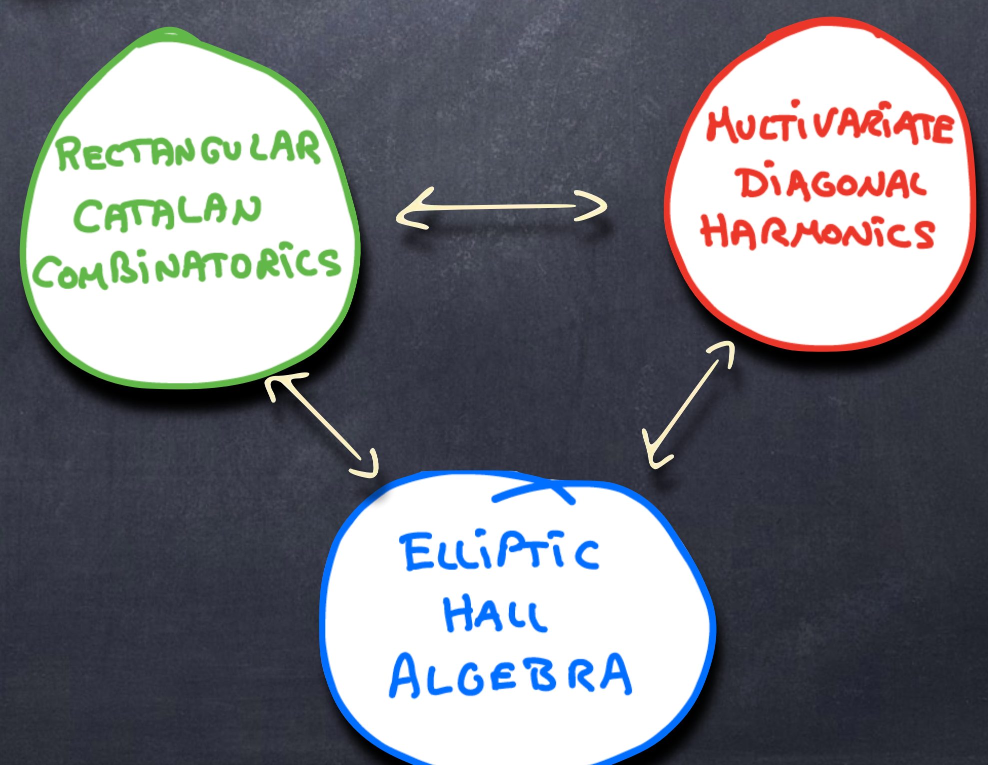
Fin



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