Algebraicity, automaticity, and invariant measures for linear cellular automata

Eric Rowland[1] & Reem Yassawi[2]

[1]Hofstra University, U.S.A & [2]Université Claude Bernard Lyon 1, France

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The cellular automata we consider are deterministic. The same local rule is applied at each cell and does not change over time.

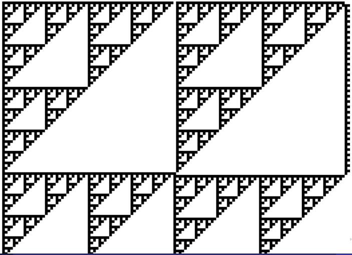
Spacetime diagrams

Example (Ledrappier)

Let $\mathcal{A} = \mathbb{F}_2$, $\phi(a,b) = a + b \mod 2$, Φ has left neighbourhood $\ell = 1$, and right neighbourhood $r = 0$. $\Phi : \mathbb{F}_2^{\mathbb{Z}} \to \mathbb{F}_2^{\mathbb{Z}}$ is surjective but not injective: $\Phi^{-1}(\cdots 00 \cdot 100 \cdots) = \{\ldots 11111 \cdot 0000 \ldots, \ldots 00000 \cdot 1111 \ldots\}$.

Definition

If $U \in \mathcal{A}^{\mathbb{Z}\times\mathbb{Z}}$ satisfies $\Phi(U|_{\mathbb{Z}\times\{n\}}) = U|_{\mathbb{Z}\times\{n+1\}}$ for each $n \in \mathbb{Z}$, we call U a *spacetime diagram* for Φ.



Measures that are invariant under the shift and a CA

Definition

The shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined as $(\sigma(x))_n := x_{n+1}$.

Question

Which probability laws μ are (σ, Φ)-invariant: $\mu = \mu \circ \sigma^{-1} = \mu \circ \Phi^{-1}$?

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Take an initial condition x, generated with a certain probability law μ, eg a Markovian law, or a Bernoulli law. This law is shift invariant. Is it invariant under the action of Φ?

Example

Let $\mu \sim (1/3, 2/3)^{\mathbb{Z}}$, so $\mu[0] = \frac{1}{3}$, $\mu[1] = \frac{2}{3}$. Let $\Phi(x) = x + \sigma^{-1}(x)$. Then $\mu \circ \Phi^{-1}[0] = \mu[00] + \mu[11] = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{5}{9} \neq \mu[0]$.

Why do we care about (σ, Φ)-invariance? Motivation

Let $f, g : [0,1] \to [0,1]$ be $f(x) = 2x \bmod 1$ and $g(x) = 3x \bmod 1$. What are the measures on $[0,1]$ such that $\mu(f^{-1}A) = \mu(A) = \mu(g^{-1}(A))$?

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Recasting in symbolic dynamics:
Let $\sigma, \Phi : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be the shift map (i.e. $\times 2$) and a cellular automaton representing $\times 3$. What are the measures such that $\mu(\sigma^{-1}A) = \mu(A) = \mu(\Phi^{-1}(A))$?

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Theorem (Rudolph, 1990)

If μ is invariant under $\times 2, \times 3$, and ergodic, and μ has positive entropy for one of $\times 2$ or $\times 3$, then μ is Lebesgue measure.

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- If $\mu \circ \Phi^{-1} \neq \mu$, does $\lim_n \mu \circ \Phi^{-n}$ exist? It would be (σ, Φ)-invariant.

Asymptotic Randomisation: heuristics

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Let $\Phi(x) = x + \sigma^{-1}(x)$. Then we identify Φ with $P(X) := 1 + X \in \mathbb{F}_p[X]$, and Φ^n is identified with $(1 + X)^n$. Then by Lucas' theorem,

$$(1 + X)^{p^n} = 1 + X^{p^n}$$

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- Work with a convergent subsequence of $\frac{1}{N}\sum_{n=0}^{N-1} \mu \circ \Phi^{-1}$: it is (σ, Φ)-invariant.

Recall that H is the probability law generated by $(1/p, 1/p, \ldots 1/p)$. It is also the Haar measure on the group ${\mathbb{F}_p}^{\mathbb{Z}}$.

Definition

We say that Φ asymptotically randomises μ if $\frac{1}{N}\sum_{n=0}^{N-1} \mu \circ \Phi^{-n} \to H$.

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Theorem (Pivato, Y, 2001+)

If μ is mixing and Markovian, and Φ is linear, then Φ asymptotically randomises μ.

Measure rigidity

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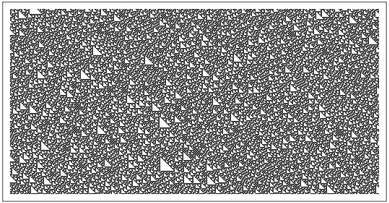
Spacetime diagram, $(1/3, 2/3)$-random initial condition, $\Phi(x) = x + \sigma(x)$



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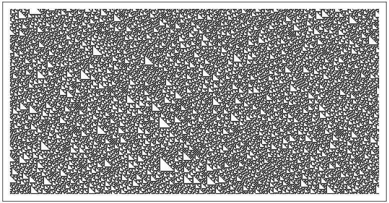

Theorem (Host, Maass, Martinez, 2003)

If μ is σ and Φ-invariant, has positive entropy for Φ and is ergodic for σ, then $\mu = H$.

Abelian cellular automata

Now consider a shift $(G^{\mathbb{Z}}, \sigma)$, where G is a finite abelian group, $G^{\mathbb{Z}}$ is a group with coordinate-wise addition and $\Phi : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ an abelian cellular automaton.

Definition

A cellular automaton $\Phi : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ is abelian if $(\Phi(v))_m = \phi_{-\ell} v_{m-\ell} + \cdots + \phi_0 v_m + \cdots + \phi_r v_{m+r}$, and each $\alpha_i : G \to G$ is a group endomorphism.

Theorem (Hellouin, Salo, Theyssier, 2018)

Let $\Phi : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be an abelian cellular automaton which has no solitons, and if μ is mixing and Markovian, then Φ asymptotically randomises μ.
There exist abelian cellular automata such that $\mu \circ \Phi^{-n} \to H$.

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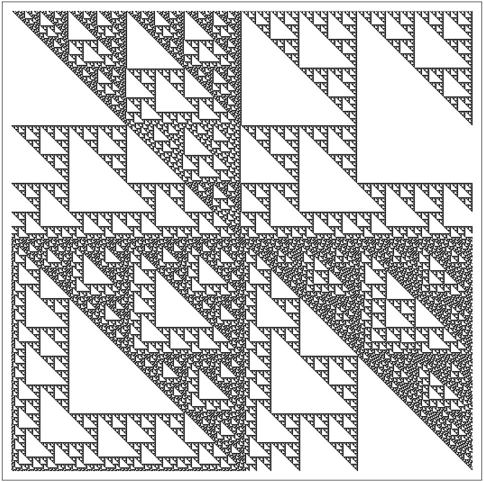
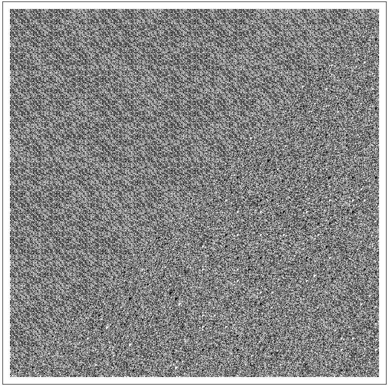
Are there measures which do not randomise? All these results are telling us to take 0-entropy initial conditions.

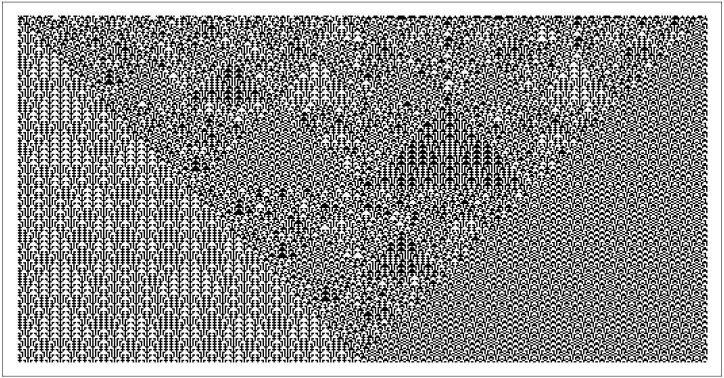
Some spacetime diagrams I



Some spacetime diagrams II



Some spacetime diagrams III

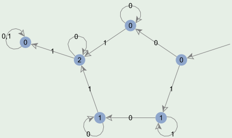


Automatic initial conditions on spacetime diagrams

Definition

A sequence $(a_n)_{n\geq 0}$ of elements in $\mathcal{A}$ is *k-automatic* if there is a DFAO $(\mathcal{S}, \Sigma_k, \delta, s_1, \mathcal{A}, \omega)$ such that $a_n = \omega(\delta(s_1, n_\ell \cdots n_1 n_0))$ for all $n \geq 0$, where $n_\ell \cdots n_1 n_0$ is the standard base-k representation of n.

Example (Catalan numbers mod 4)



The 2-automatic sequence produced by this automaton is

$$(a_n)_{n\geq 0} = 0, 1, 2, 1, 2, 2, 0, 1, 2, 2, 0, 2, 0, 0, 0, 1, \ldots.$$

Definition

1. A sequence $(u_m)_{m\in\mathbb{Z}}$ is $(-p)$-automatic if the sequences $(u_m)_{m\geq 0}$ and $(u_{-m})_{m\geq 0}$ are p-automatic.
2. The sequence $(U_{m,n})_{m\geq 0, n\geq 0} \in \mathbb{F}_p^{\mathbb{N}\times\mathbb{N}}$ is $[p,p]$-automatic if there is a DFAO $(\mathcal{S}, \{0,\dots,p-1\}^2, \delta, s_0, \mathbb{F}_p, \omega)$ such that

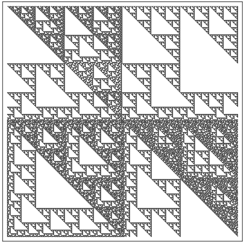
$$U_{m,n} = \omega(\delta(s_0, (m_\ell, n_\ell)\cdots(m_1,n_1)(m_0,n_0)))$$

for all $(m,n)\in\mathbb{N}\times\mathbb{N}$, where $m_\ell\cdots m_1 m_0$ is a base-p representation of m and $n_\ell\cdots n_1 n_0$ is a base-p representation of n. Here, if m and n have standard base-p representations of different lengths, then the shorter representation is padded with leading zeros.
3. A sequence $U\in\mathbb{F}_p^{\mathbb{Z}\times\mathbb{Z}}$ is $[-p,-p]$-automatic if each of $U|_{(\pm\mathbb{N})\times(\pm\mathbb{N})}$ is $[p,p]$-automatic.

Automatic spacetime diagrams

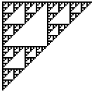
Theorem (Rowland-Y, 2018)

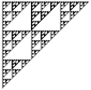
Let $\Phi : \mathbb{F}_p^{\mathbb{Z}} \to \mathbb{F}_p^{\mathbb{Z}}$ be a linear cellular automaton with left and right radii ℓ and r. Let $U \in \mathbb{F}_p^{\mathbb{Z}\times\mathbb{Z}}$ be a spacetime diagram for Φ. If the initial conditions that define U are a collection of p-automatic sequences, then U is $[-p, -p]$-automatic.



Spacetime diagram for the Ledrappier CA, where the initial conditions are the Thue-Morse sequence on the +ve horizontal axis, the reflection of the TM sequence on both the -ve horizontal axis and the -ve vertical axis.

Previous work and main tool





Theorem (Allouche, von Haeseler, Lange, Petersen, Peitgen, & Skordev, 96-97)

If $u \in (\mathbb{Z}/n\mathbb{Z})^{\mathbb{N}}$ is finite, and the LCA whose generating polynomial $\Phi \in \mathbb{Z}/n\mathbb{Z}[X]$ is not a monomial, then the spacetime diagram $U \in (\mathbb{Z}/n\mathbb{Z})^{\mathbb{N}\times\mathbb{N}}$ is $[n, n]$-automatic if and only if $n = p^k$.

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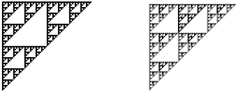

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Main tool:

Theorem (Christol's theorem)

1. *A sequence $(u_m)_{m\geq 0}$ of elements in $\mathbb{F}_p$ is p-automatic if and only if $\sum_{m\geq 0} u_m x^m$ is algebraic over $\mathbb{F}_p(x)$.*
2. *A sequence of elements $(U_{m,n})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ in $\mathbb{F}_p$ is $[p, p]$-automatic if and only if $\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} U_{m,n}x^m y^n$ is algebraic over $\mathbb{F}_p(x, y)$.*

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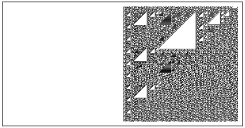

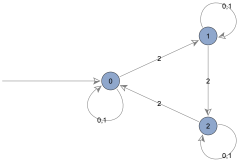

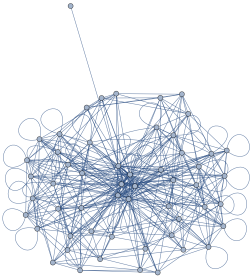

Transfer principle: (σ, Φ)-invariant measures from spacetime diagrams

Let $U \in \mathbb{F}_p^{\mathbb{Z}\times\mathbb{Z}}$ be a spacetime diagram for the LCA Φ. Define

$$X_U := \{V \in \mathbb{F}_p^{\mathbb{Z}\times\mathbb{Z}} : \mathcal{L}_V \subseteq \mathcal{L}_U\},$$

and let σ_1 and σ_2 denote the horizontal and vertical shift.

Lemma

Every element of X_U is a spacetime diagram for Φ.
There exist (σ_1, σ_2)-invariant measures supported on X_U, and each such measure μ yields a (σ, Φ)-invariant measure $\pi\mu$.

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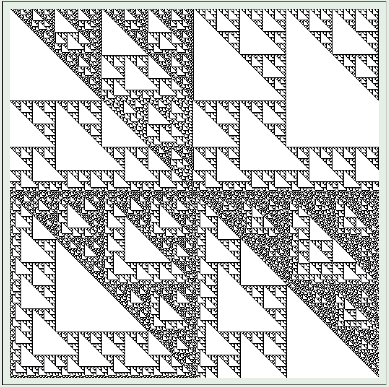
So we have some (σ, Φ)-invariant measures. But are they H, or "trivial"?

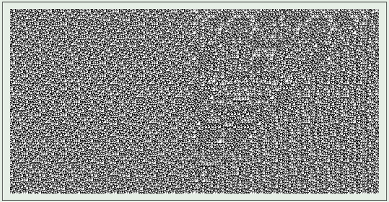
Proposition (after Berthé, Allouche/Shallit)

If the sequence $U \in \mathbb{F}_p^{\mathbb{Z}\times\mathbb{Z}}$ is $[-p, -p]$-automatic, then for some K, its complexity function satisfies $c_U(m, n) \leq K \max\{m, n\}^{10}$, so that if μ is a (σ_1, σ_2)-invariant measure, then $\pi\mu \neq H$.

Examples

Example





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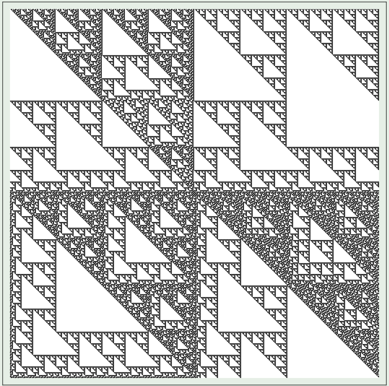

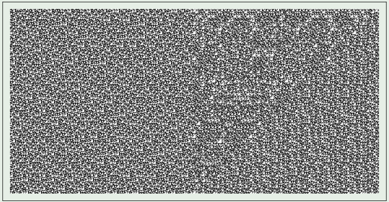

The only (σ_1, σ_2)-invariant measures supported on the left diagram is the Dirac mass supported on the constant 0 configuration. However there are nontrivial measures supported on the right diagram.

Nontrivial jointly invariant substitutional measures

Theorem (Cobham's theorem)

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Given a substitution $\theta : \mathbb{F}_p \to \mathbb{F}_p^p$, we write $\theta(a) = \theta_0(a) \cdots \theta_{p-1}(a)$. We say that θ is bijective if, for each $0 \leq i \leq p-1$, $\{\theta_i(a) : a \in \mathbb{F}_p\} = \mathbb{F}_p$.

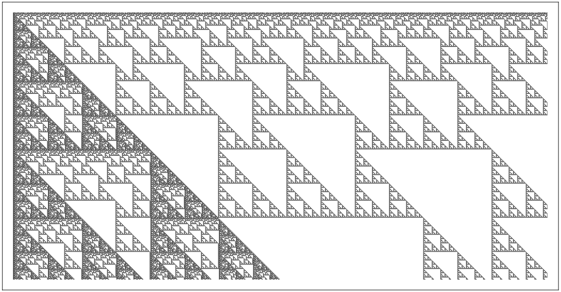
Example

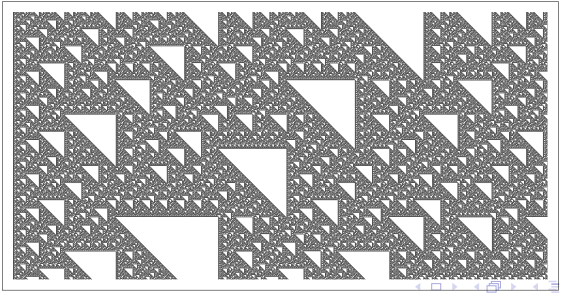
$\theta(0) = 001$, $\theta(1) = 112$, and $\theta(2) = 220$ is bijective. If $W \subset \mathbb{F}_p^p$ is a word containing all letters, then $\theta(i) := w + i^p$ is bijective. We call θ a *rotation*.

Theorem (Rowland-Y, 2018)

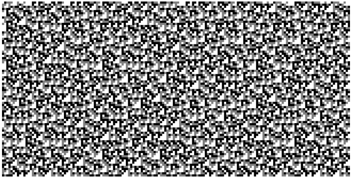
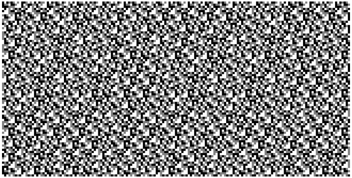
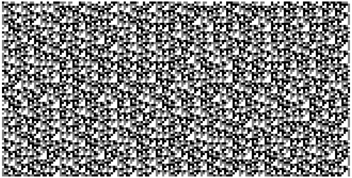
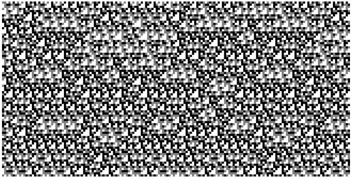
Let $\Phi : \mathbb{F}_p^{\mathbb{Z}} \to \mathbb{F}_p^{\mathbb{Z}}$ be a linear cellular automaton whose generating polynomial has L terms, and let $u \in \mathbb{F}_p^{\mathbb{Z}}$ be a θ-fixed point of a rotational substitution. If p is not a divisor of L, $\mathrm{ST}_\Phi(u)$ supports nontrivial (σ, Φ)-invariant measures.

Dependance on p: Φ Ledrappier & $p = 2$





Dependance on p: Φ Ledrappier & $p = 3$

Conclusions

- We have new nontrivial measures that are jointly (σ, Φ) invariant. We suspect that there are many, but we need conditions on Φ and the initial conditions which would guarantee uniform recurrence of the STDs we generate.
- What is the relationship between our measures and the measures arising from Intersection sets as defined by Kitchens, Schmidt, Einsiedler?