Résolution d'équations logiques

Philippe Balbiani

CNRS — Toulouse University

Institut de recherche en informatique de Toulouse

Institut de Recherche en Informatique de Toulouse
CNRS - INP - UT3 - UT1 - UT2J

Introduction

Unification problem in a logical system L

- Given a formula $\psi(x_1, \ldots, x_n)$
- Determine whether there exists formulas φ_1, ..., φ_n such that $\psi(\varphi_1, \ldots, \varphi_n)$ is in L

Admissibility problem in a logical system L

- Given a rule of inference $\frac{\varphi_1(x_1,\ldots,x_n),\ \ldots,\ \varphi_m(x_1,\ldots,x_n)}{\psi(x_1,\ldots,x_n)}$
- Determine whether for all formulas χ_1, ..., χ_n, if $\varphi_1(\chi_1, \ldots, \chi_n)$, ..., $\varphi_m(\chi_1, \ldots, \chi_n)$ are in L then $\psi(\chi_1, \ldots, \chi_n)$ is in L

Algebraic semantics of classical propositional logic

Boolean algebras $(A, 0_A, 1_A, -_A, \cup_A, \cap_A)$

- Given a finite set $\{(\varphi_i, \psi_i) : \ i = 1 \dots n\}$ of pairs of formulas
- Determine if there exists a substitution σ such that
  - $\models_{BA} \sigma(\varphi_i) = \sigma(\psi_i)$ for all $i = 1 \dots n$
  - $\models_{BA} \sigma(\varphi_i) \leftrightarrow \sigma(\psi_i)$ for all $i = 1 \dots n$

Algebraic semantics of intuitionistic propositional logic

Heyting algebras $(A, 0_A, 1_A, \cup_A, \cap_A, \rightarrow_A)$

- Given a finite set $\{(\varphi_i, \psi_i) : \ i = 1 \dots n\}$ of pairs of formulas
- Determine if there exists a substitution σ such that
  - $\models_{HA} \sigma(\varphi_i) = \sigma(\psi_i)$ for all $i = 1 \dots n$
  - $\models_{HA} \sigma(\varphi_i) \leftrightarrow \sigma(\psi_i)$ for all $i = 1 \dots n$

Introduction

Why unification?

Description logic

Given concept definitions $C(x_1, \ldots, x_n)$ and $D(x_1, \ldots, x_n)$

- Determine whether there are some redundancies between $C(x_1, \ldots, x_n)$ and $D(x_1, \ldots, x_n)$
- Solve $C(x_1, \ldots, x_n) \equiv D(x_1, \ldots, x_n)$

Epistemic planning

Given variable-free epistemic formulas $\varphi(p_1, \ldots, p_m)$ and $\psi(p_1, \ldots, p_m)$

- Determine whether there exists a public announcement χ such that $\models \varphi \to \langle \chi ! \rangle \psi$
- Solve $\models \varphi \to \langle x ! \rangle \psi$

Introduction

If L is consistent then the following are equivalent:

- Formula $\varphi(x_1, \dots, x_n)$ is unifiable
- Rule $\frac{\varphi(x_1,\dots,x_n)}{\bot}$ is non-admissible

If L is finitary then the following are equivalent:

- Rule $\frac{\varphi_1(x_1,\dots,x_n),\dots,\varphi_m(x_1,\dots,x_n)}{\psi(x_1,\dots,x_n)}$ is admissible
- Formulas $\psi(\chi_1, \dots, \chi_n)$ is in L for each maximal unifiers $(\chi_1, \dots, \chi_n)$ of formulas $\varphi_1(x_1, \dots, x_n), \dots, \varphi_m(x_1, \dots, x_n)$

Introduction

Unification: some examples

- The formula $(x \to p) \wedge (q \to y)$ is unifiable in CPL
- The formula $\Box\neg x \vee \Box x$ is unifiable in modal logic K

In Classical Logic

- Unification **is equivalent to** satisfiability
- Why ? Use the inference rule of Uniform Substitution

In Modal Logic

- Unification in $S4$, $S5$, etc **is not equivalent to** satisfiability
- Why ? Consider the formula $\Diamond x \wedge \Diamond\neg x$ and use the inference rule of Uniform Substitution

Introduction

About Classical Propositional Logic

Classical Propositional Logic is **structurally complete**

- Thus, admissibility in Classical Propositional Logic is decidable

About intermediate logics

Rybakov (1981): If L is an intermediate logic then the following are equivalent

- Rule $\mathcal{R}$ is admissible in L
- The modal translation of rule $\mathcal{R}$ is admissible in the greatest modal companion of L

Introduction

Rybakov (1982)

- The admissibility problem in extensions of $S4.3$ is decidable

Rybakov (1984)

- The admissibility problem in $S4$ is decidable

Chagrov (1992)

- There exists a decidable normal modal logic with an undecidable admissibility problem

Wolter and Zakharyaschev (2008)

- The unification problem for any normal modal logic between K_U and $K4_U$ is undecidable

Introduction

Contents

- **Definitions**
- Boolean unification
- Modal unification
- Unification types in modal logics
- Recent advances

Definitions

Let L be a propositional logic

Substitutions

- σ: variable $x \mapsto$ formula $\sigma(x)$

Applying substitutions to formulas

- $\sigma(\varphi(x_1, \ldots, x_n)) = \varphi(\sigma(x_1), \ldots, \sigma(x_n))$

Composition of substitutions

- $\sigma \circ \tau$: variable $x \mapsto$ formula $\tau(\sigma(x))$

Equivalence relation between substitutions

- $\sigma \simeq_L \tau$ iff for all variables x, $\sigma(x) \leftrightarrow \tau(x) \in L$
- "σ and τ are L-equivalent"

Partial order between substitutions

- $\sigma \preceq_L \tau$ iff there exists a substitution μ such that $\sigma \circ \mu \simeq_L \tau$
- "σ is less specific, more general than τ in L"

Definitions

Let L be a propositional logic

Unifiers

- A substitution σ is a unifier of a formula φ iff $\sigma(\varphi) \in L$

Complete sets of unifiers

- A set Σ of unifiers of a formula φ is complete iff for all unifiers τ of φ, there exists a unifier σ of φ in Σ such that $\sigma \preceq_L \tau$

Important questions

- Given a formula, has it a unifier?
- If so, has it a minimal complete set of unifiers?
- If so, how large is this set? Is this set effectively calculable?

- Definitions
- **Boolean unification**
- Modal unification
- Unification types in modal logics
- Unification in description logics
- Recent advances

Boolean unification

Syntax

- $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi)$

Abbreviations for $\top$, $\wedge$, etc

- As usual

Examples of Boolean unification problems

- $(x \leftrightarrow y) \leftrightarrow (x \vee y)$
- $(x \rightarrow y) \wedge (\neg x \rightarrow z)$
- $(x \rightarrow p) \wedge (q \rightarrow y)$

Boolean unification

Proposition:

Without parameters, Boolean unification **is *NP*-complete**

- $\varphi(\bar{x})$ is *CPL*-unifiable $\iff \exists\bar{x}\varphi(\bar{x})$ is *QBF*-valid

With parameters, Boolean unification **is Π_2^P-complete**

- $\varphi(\bar{p},\bar{x})$ is *CPL*-unifiable $\iff \forall\bar{p}\exists\bar{x}\varphi(\bar{p},\bar{x})$ is *QBF*-valid

Baader (1998)

Boolean unification

Projective formulas

- A formula φ is said to be **projective** iff it has a unifier σ such that $\varphi \to (\sigma(x) \leftrightarrow x)$ is in *CPL*

Any unifier σ of φ satisfying the above condition is called a **projective unifier** of φ

Lemma: Projective unifiers **are closed under compositions**

Lemma: Projective unifiers **are most general unifiers**

Boolean unification

Lemma: Unifiable formulas are projective
Proof: Consider a unifier σ of φ

- Let ϵ be the substitution such that $\epsilon(x) = (\varphi \wedge x) \vee (\neg\varphi \wedge \sigma(x))$
- **Fact:** ϵ is a projective unifier of φ

Proposition: Boolean unification **is unitary,** i.e. every unifiable formula has a most general unifier

Remarks about ϵ

- ϵ is the so-called “Löwenheim substitution”
- If σ is atom-free then ϵ can be defined by
  - $\epsilon(x) = \varphi \wedge x$ when $\sigma(x) = \bot$
  - $\epsilon(x) = \varphi \rightarrow x$ when $\sigma(x) = \top$

- Definitions
- Boolean unification
- **Modal unification**
- Unification types in modal logics
- Unification in description logics
- Recent advances

Modal unification

Syntax

- $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$

Abbreviation

- $\Diamond\varphi ::= \neg\Box\neg\varphi$

Examples of modal unification problems

- $\Box\neg x \vee \Box x$
- $x \to \Box x$
- $(x \to p) \wedge (x \to \Box(p \to x))$

Modal unification

Semantics

- **Frame:** directed graph $\mathcal{F} = (W, R)$
- **Models:** $\mathcal{M} = (W, R, V)$ where $V \colon x, p \mapsto V(x), V(p) \subseteq W$

Truth conditions in a model

- $\mathcal{M}, s \models x$ iff $s \in V(x)$ and $\mathcal{M}, s \models p$ iff $s \in V(p)$
- $\mathcal{M}, s \models \Box\varphi$ iff $\forall t \in W$, if sRt then $\mathcal{M}, t \models \varphi$

Validity in a frame

- φ is **valid** in frame $\mathcal{F}$ iff φ is true at every node of every model based on $\mathcal{F}$

Normal modal logic L determined by a class $\mathcal{C}$ of frames

- Set of all formulas that are valid in the frames of $\mathcal{C}$

Modal unification

Lemma: The unification problem **is trivially decidable (*NP*-complete)** for any normal modal logic containing $\Diamond\top$

- *KD*, *KT*, *S*4, *S*4.3, *S*5

— **Rybakov 1984, 1997:** The unification and admissibility problems **are decidable** for intuitionistic logic, *GL* and *S*4
— **Jeřábek 2005, 2007:** The admissibility problem **is *coNEXPTIME*-complete** for intuitionistic logic, *GL* and *S*4
— **Chagrov 1992:** Only one — rather artificial — example of a **decidable** unimodal logic for which the admissibility problem is **undecidable**
— **Wolter and Zakharyaschev 2008:** The unification problem for modal logics between K_u and $K4_u$ **is undecidable**

Modal unification

The unification and admissibility problems for K itself . . .

- . . . still remain **open**

Nothing is known about

- **The decidability status** of the unification and admissibility problems for
  - Basic modal logic K
  - Various multimodal logics
  - Various hybrid logics
  - Various description logics

- Definitions
- Boolean unification
- Modal unification
- **Unification types in modal logics**
- Recent advances

Let L be a propositional logic and φ be a formula

An ***L*-unifier** of φ is a substitution σ such that

- $\sigma(\varphi) \in L$

We shall say that φ is of **type unitary (1)** for L iff

- There exists a complete minimal set Σ of L-unifiers of φ
- $Card(\Sigma) = 1$

Example in *CPL*: $(x \to p) \wedge (q \to y)$ is unitary

- $\sigma(x) = p \wedge x$ and $\sigma(y) = q \vee y$

Let L be a propositional logic and φ be a formula

An ***L*-unifier** of φ is a substitution σ such that

- $\sigma(\varphi) \in L$

We shall say that φ is of **type finitary (ω)** for L iff

- There exists a complete minimal set Σ of L-unifiers of φ
- $Card(\Sigma) \neq 1$ but Σ is finite

Example in IPL: $x \vee \neg x$ is finitary

- $\sigma(x) = \top$
- $\tau(x) = \bot$

Let L be a propositional logic and φ be a formula

An ***L*-unifier** of φ is a substitution σ such that

- $\sigma(\varphi) \in L$

We shall say that φ is of **type infinitary (∞)** for L iff

- There exists a complete minimal set Σ of L-unifiers of φ
- Σ is infinite

No known example of an infinitary formula in modal logics

Let L be a propositional logic and φ be a formula

An **L-unifier** of φ is a substitution σ such that

- $\sigma(\varphi) \in L$

We shall say that φ is of **type nullary (0)** for L iff

- There exists no complete minimal set of L-unifiers of φ

Example in K: $x \to \Box x$ is nullary

- $\sigma_\top(x) = \top$
- $\sigma_k(x) = \Box^{<k} x \wedge \Box\bot$

Let L be a propositional logic and φ be a formula

An **L-unifier** of φ is a substitution σ such that

- $\sigma(\varphi) \in L$

We shall say that φ is of **type nullary (0)** for L iff

- There exists no complete minimal set of L-unifiers of φ

Example in K: $\Box\neg x \vee \Box x$ is finitary

- $\sigma(x) = \top$
- $\tau(x) = \bot$

Let L be a propositional logic

We shall say that L is of **type unitary** iff

- Every L-unifiable formula is **unitary**

We shall say that L is of **type finitary** iff

- Every L-unifiable formula is **unitary or finitary**
- There are finitary L-unifiable formulas

Examples

- Unification in **classical propositional logic is unitary**
- Unification in **intuitionistic propositional logic is finitary**

Let L be a propositional logic

We shall say that L is of **type infinitary** iff

- Every L-unifiable formula is **unitary or finitary or infinitary**
- There are infinitary L-unifiable formulas

We shall say that L is of **type nullary** iff

- There are nullary L-unifiable formulas

Example

- No known example of an infinitary modal logic
- Unification in **modal logic** K **is nullary**

Modal logic $S5$

- Syntax
  - $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$
- Abbreviations
  - $\Diamond\varphi ::= \neg\Box\neg\varphi$
- Semantics
  - **Frame:** partition $\mathcal{F} = (W, R)$, i.e. R is an equivalence relation
  - **Model:** $\mathcal{M} = (W, R, V)$ where V: $x, p \mapsto V(x), V(p) \subseteq W$
- **Truth conditions** in a model $\mathcal{M} = (W, R, V)$
  - $\mathcal{M}, s \models x$ iff $s \in V(x)$ and $\mathcal{M}, s \models p$ iff $s \in V(p)$
  - $\mathcal{M}, s \models \Box\varphi$ iff $\forall t \in W$, if sRt then $\mathcal{M}, t \models \varphi$

Projective formulas

- A formula φ is said to be **projective** iff it has a unifier σ such that $\Box\varphi \to (\sigma(x) \leftrightarrow x)$ is in S5

Any unifier σ of φ satisfying the above condition is called a **projective unifier** of φ

Lemma Projective unifiers **are closed under compositions**

Lemma Projective unifiers **are most general unifiers**

Unification types in modal logics

Unification in *S*5

Lemma Unifiable formulas **are projective**

Proof: Consider a unifier σ of φ

- Let ϵ be the substitution such that $\epsilon(x) = (\Box\varphi \wedge x) \vee (\neg\Box\varphi \wedge \sigma(x))$
- **Fact:** ϵ is a projective unifier of φ

Proposition *S*5 unification **is unitary,** i.e. every unifiable formula has a most general unifier

Remarks about ϵ

- ϵ is the **Löwenheim substitution**
- If σ is atom-free then ϵ can be defined by
  - $\epsilon(x) = \Box\varphi \wedge x$ when $\sigma(x) = \bot$
  - $\epsilon(x) = \Box\varphi \rightarrow x$ when $\sigma(x) = \top$

Unification types in modal logics

Unification in $K4$

Modal logic $K4$

- Syntax
  - $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$
- Abbreviations
  - $\Diamond\varphi ::= \neg\Box\neg\varphi$
  - $\Box^{+}\varphi ::= \varphi \wedge \Box\varphi$
- Semantics
  - **Frame:** directed graph $\mathcal{F} = (W, R)$ where R is **transitive**
  - **Model:** $\mathcal{M} = (W, R, V)$ where V: $x, p \mapsto V(x), V(p) \subseteq W$
- **Truth conditions** in a model $\mathcal{M} = (W, R, V)$
  - $\mathcal{M}, s \models x$ iff $s \in V(x)$ and $\mathcal{M}, s \models p$ iff $s \in V(p)$
  - $\mathcal{M}, s \models \Box\varphi$ iff $\forall t \in W$, if sRt then $\mathcal{M}, t \models \varphi$

Unification types in modal logics

Unification in $K4$

Proposition (Rybakov 1984, 1997) $K4$-unification **is decidable**

Proposition (Ghilardi 2000) $K4$-unification **is finitary,** i.e.

- For all formulas $\varphi(x_1, \ldots, x_n)$, the cardinality of a minimal complete set of $K4$-unifiers **is finite**

Unification types in modal logics

Unification in $K4$

Ghilardi (2000): A formula $\varphi(x_1, \ldots, x_n)$ is said to be **projective** iff **there exists a substitution** σ such that

1. σ is a $K4$-unifier of φ
2. $\Box^+\varphi \to (x_i \leftrightarrow \sigma(x_i)) \in K4$ for each i such that $1 \leq i \leq n$

Wroński (1995): A formula $\varphi(x_1, \ldots, x_n)$ is said to be **transparent** iff **there exists a substitution** σ such that

1. σ is a $K4$-unifier of φ
2. for all $K4$-unifiers τ of φ, $\tau(x_i) \leftrightarrow \tau(\sigma(x_i)) \in K4$ for each i such that $1 \leq i \leq n$

Ghilardi (2000): A formula $\varphi(x_1, \ldots, x_n)$ is said to be **projective** iff **there exists a substitution** σ such that

1. σ is a $K4$-unifier of φ
2. $\Box^+\varphi \to (x_i \leftrightarrow \sigma(x_i)) \in K4$ for each i such that $1 \leq i \leq n$

For all $A \subseteq \{1, \ldots, n\}$, let θ_φ^A be the substitution defined by

- $\theta_\varphi^A(x_i) = \Box^+\varphi \wedge x_i$ if $i \notin A$
- $\theta_\varphi^A(x_i) = \Box^+\varphi \to x_i$ if $i \in A$

Remark: The substitution θ_φ^A **satisfies condition** 2

Ghilardi (2000): A formula $\varphi(x_1, \ldots, x_n)$ is said to be **projective** iff **there exists a substitution** σ such that

1. σ is a $K4$-unifier of φ
2. $\Box^+\varphi \to (x_i \leftrightarrow \sigma(x_i)) \in K4$ for each i such that $1 \leq i \leq n$

For all $A \subseteq \{1, \ldots, n\}$, let θ_φ^A be the substitution defined by

- $\theta_\varphi^A(x_i) = \Box^+\varphi \to x_i$ if $i \in A$
- $\theta_\varphi^A(x_i) = \Box^+\varphi \wedge x_i$ if $i \notin A$

Given an arbitrary enumeration $A_1, \ldots, A_{2^n}$ of the subsets of $\{1, \ldots, n\}$, let $\theta_\varphi = \theta_\varphi^{A_1} \circ \ldots \circ \theta_\varphi^{A_{2^n}}$

Proposition For all formulas $\varphi(x_1, \ldots, x_n)$, if $d = depth(\varphi)$ and N is the number of non-$\sim_d$-equivalent models over $x_1, \ldots, x_n$, the following statements are equivalent:

- ${\theta_\varphi}^{2N}$ is a $K4$-unifier of φ
- φ is projective

- **Ghilardi, S.:** *Best solving modal equations.* Annals of Pure and Applied Logic **102** (2000) 183–198.

Corollary It is decidable to determine whether a given formula φ is projective

Lemma For all formulas φ and for all substitutions σ, if σ is a $K4$-unifier of φ

- There exists a formula ψ, $depth(\psi) \leq depth(\varphi)$, such that
  - ψ is projective
  - σ is a $K4$-unifier of ψ
  - $\Box^{+}\psi \to \varphi \in K4$

Proposition (Ghilardi 2000) $K4$-unification **is finitary,** i.e.

- For all formulas $\varphi(x_1, \ldots, x_n)$, the cardinality of a minimal complete set of $K4$-unifiers **is finite**

Modal logic K

- Syntax
  - $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$
- Abbreviations
  - $\Diamond\varphi ::= \neg\Box\neg\varphi$
  - $\Box^{<n}\varphi ::= \Box^0\varphi \wedge \ldots \wedge \Box^{n-1}\varphi$ for each $n \in \mathcal{N}$
- Semantics
  - **Frame:** directed graph $\mathcal{F} = (W, R)$
  - **Model:** $\mathcal{M} = (W, R, V)$ where $V\colon x, p \mapsto V(x), V(p) \subseteq W$
- **Truth conditions** in a model $\mathcal{M} = (W, R, V)$
  - $\mathcal{M}, s \models x$ iff $s \in V(x)$ and $\mathcal{M}, s \models p$ iff $s \in V(p)$
  - $\mathcal{M}, s \models \Box\varphi$ iff $\forall t \in W$, if sRt then $\mathcal{M}, t \models \varphi$

Open question **Is K-unification decidable?**

K-unification **is not unitary** since

- $\sigma_{\top}(x) = \top$ and $\sigma_{\bot}(x) = \bot$ constitute a minimal complete set of unifiers in K of the formula $\Box\neg x \vee \Box x$

K-unification **is nullary,** i.e.

- There exists a formula φ such that there exists no complete minimal set of K-unifiers of φ

Method (Jeřábek, 2014) Study the K-unifiers of

- $x \to \Box x$

Consider the following substitutions

- $\sigma_n(x) = \Box^{<n} x \wedge \Box^n \bot$ for each $n \in \mathcal{N}$
- $\sigma_\top(x) = \top$

Lemma

- σ_n **is a K-unifier** of $x \to \Box x$ for each $n \in \mathcal{N}$
- $\sigma_\top$ **is a K-unifier** of $x \to \Box x$

Lemma

- $\sigma_n \preceq_K \sigma_m$ iff $m \leq n$

Unification types in modal logics

Unification in K

Proposition (Jeřábek, 2014) For all formulas φ, $depth(\varphi) = n$,

- If $\varphi \rightarrow \Box\varphi \in K$ then either $\varphi \rightarrow \Box^n\bot \in K$, or $\varphi \in K$

Corollary The following substitutions form a complete set of K-unifiers for the formula $x \rightarrow \Box x$

- $\sigma_n(x) = \Box^{<n}x \wedge \Box^n\bot$ for each $n \in \mathcal{N}$
- $\sigma_\top(x) = \top$

Corollary K-unification **is nullary**

Intuitionistic propositional logic — *IPL*

Ghilardi (1999): for every *IPL*-unifiable formula φ, one can find a finite number of projective ψ_i such that

- $\psi_i \to \varphi$ is in *IPL*
- every *IPL*-unifier for φ is also an *IPL*-unifier for one of the ψ_i

Logic of weak excluded middle — *KC*

Ghilardi (1999):

- *KC* is unitary
- *KC* is the least intermediate logic having unitary unification

Logic of Gödel and Dummett — *LC*

Wroński (2008):

- An intermediate logic L has projective unification iff $LC \subseteq L$

Extensions of *S*4

Dzik and Wojtylak (2011):

- In all extensions of $S4.3$, unifiable formulas have projective unifiers
- Extensions of $S4$ in which all unifiable formulas have projective unifiers must contain $S4.3$

Extensions of $K4$

Ghilardi and Sacchetti (2004):

- Define the abbreviations
  - $\Box^+\varphi := (\Box\varphi \wedge \varphi)$
  - $\Diamond^+\varphi := (\Diamond\varphi \vee \varphi)$
- $K4.2^+$ is $K4 + \Diamond^+\Box^+\varphi \rightarrow \Box^+\Diamond^+\varphi$
- An extension L of $K4$ has filtering unification iff $K4.2^+ \subseteq L$

Splitting pair $(L(f_2), S4.2)$

Dzik (2006): For all extensions L of $S4$

- Either $L \subseteq L(f_2)$ or $S4.2 \subseteq L$
- If $L \subseteq L(f_2)$ then L is not unitary
- If $S4.2 \subseteq L$ then L is unitary or nullary

- Definitions
- Boolean unification
- Modal unification
- Unification types in modal logics
- **Recent advances**

Recent advances

$$KD = K + \Diamond\top$$

KD is nullary

- $(x \to p) \land (x \to \Box(p \to x))$

B. and Gencer (2018)

$$KT = K + \Box\varphi \to \varphi$$

KT is nullary

- $(x \to p) \land (x \to \Box(q \to y)) \land (y \to q) \land (y \to \Box(p \to x))$

B. (to appear)

$$KB = K + \varphi \to \Box\Diamond\varphi$$

KB is nullary

- $x \to (\neg p \land \neg q \to \Box(p \land \neg q \to \Box(\neg p \land q \to \Box(\neg p \land \neg q \to x))))$

B. and Gencer (submitted for publication)

Recent advances

$Alt_1 \;=\; K + \Diamond\varphi \to \Box\varphi$

- Alt_1 is nullary for unification
- The unification problem (without parameters) in Alt_1 is decidable (in *PSPACE*)

B. and Tinchev (2016)

Normal extensions of $K5 \;=\; K + \Diamond\varphi \to \Box\Diamond\varphi$

- These modal logics are unitary for unification

$K + \Box^k\bot$ for $k \geq 2$

- These modal logics are finitary for unification

B., Rostamigiv and Tinchev (submitted for publication)

Recent advances

Unification in Dynamic Epistemic Logics

Syntax

- $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid K_a\varphi \mid [\varphi!]\psi$

Abbreviations

- $\hat{K}_a\varphi ::= \neg K_a \neg\varphi$
- $\langle\varphi!\rangle\psi ::= \neg[\varphi!]\neg\psi$

Readings

- $K_a\varphi$: “agent *a* knows that φ holds”
- $[\varphi!]\psi$: “if φ holds then ψ will hold after φ is announced”
- $\hat{K}_a\varphi$: “it is compatible with *a*’s knowledge that φ holds”
- $\langle\varphi!\rangle\psi$: “$\varphi$ holds and ψ will hold after φ is announced”

Recent advances

A simple example of unification problem

Public announcements : $K_a, K_b, \ldots$ are $S5$ modalities

- $P_1 \; = \; \varphi \rightarrow \langle x! \rangle K_a \psi$ — with ψ Boolean formula
- $P_2 \; = \; (\varphi \rightarrow x) \wedge (\varphi \rightarrow K_a(x \rightarrow \psi))$
- $P_3 \; = \; (\varphi \rightarrow x) \wedge (\hat{K}_a \varphi \rightarrow (x \rightarrow \psi))$
- $P_4 \; = \; (\varphi \rightarrow x) \wedge (x \rightarrow (\hat{K}_a \varphi \rightarrow \psi))$
- Necessary condition: $\models \varphi \rightarrow (\hat{K}_a \varphi \rightarrow \psi)$, i.e. $\models \varphi \rightarrow \psi$
- Unifier of P_4:
  - $\sigma(x) = \psi$
- Most general unifier of P_4:
  - $\epsilon(x) = (P_4 \wedge x) \vee (\neg P_4 \wedge \sigma(x))$
  - $\epsilon(x) = ((\hat{K}_a \varphi \rightarrow \psi) \wedge x) \vee (\varphi \wedge \neg x)$

Recent advances

Other examples of unification problems

- $\varphi \to \langle x! \rangle K_a \psi$
- $\varphi \to \langle x! \rangle (K_{a_1} \psi_1 \wedge \ldots \wedge K_{a_n} \psi_n)$
- $\varphi \to \langle x! \rangle K_{a_1} \ldots K_{a_n} \psi$
- $\varphi \to \langle K_b x! \rangle K_a \psi$
- $\varphi \to \langle K_b x! \rangle (K_{a_1} K_b \psi_1 \wedge \ldots \wedge K_{a_1} \hat{K}_b \chi_1 \wedge \ldots)$

Recent advances

A simple example of unification problem

Lies : $K_a, K_b, \ldots$ are $KD45$ modalities

- $P_1 \; = \; \varphi \rightarrow \langle x! \rangle K_a \psi$ — with ψ Boolean formula
- $P_2 \; = \; (\varphi \rightarrow \neg x) \wedge (\varphi \rightarrow K_a(x \rightarrow \psi))$
- Unifier of P_2:
- $\sigma(x) = \bot$
- Most general unifier of P_2:
  - $\epsilon(x) = ???$

Conclusion

Some open problems

Decidability of

- parameter-free unification in modal logic K, KB ?
- unification with parameters in modal logics KD, KDB ?
- unification with parameters in modal logics KT, KTB ?
- unification with parameters in modal logics Alt_1, Alt_2 ?
- unification in implication fragments ?

Type of

- KB, KD, KDB, KT, KTB for parameter-free unification ?
- $S5 \otimes S5$ and other fusions of modal logics ?
- $S4.2 \times S4.2$ and other products of modal logics ?
- $K + \Box^k \bot$ and other locally tabular modal logics ?
- unification in implication fragments ?

Thank you!
Questions?