La théorie des jeux en synthèse assistée par ordinateur

Véronique Bruyère
UMONS Belgium

GDR 2019

1 Topic

2 Qualitative two-player zero-sum games

3 Quantitative two-player zero-sum games

4 Two-player zero-sum games: several extensions

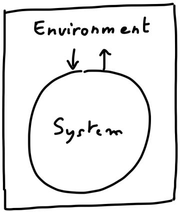
5 Multiplayer non zero-sum games

6 Conclusion

Topic

Reactive systems

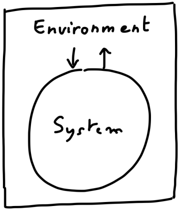
- System embedded into an uncontrollable environment
- It must satisfy some property against any behavior of the environment
- How to automatically design a correct controller for the system?



Topic

Reactive systems

- System embedded into an uncontrollable environment
- It must satisfy some property against any behavior of the environment
- How to automatically design a correct controller for the system?



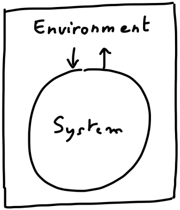
Example

- System: autonomous robotized lawnmower
  Environment: weather, cat
- The lawnmower must cut the grass in any conditions
- How to design a correct lawnmower controller?

Topic

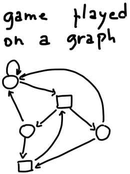
Reactive systems

- System embedded into an uncontrollable environment
- It must satisfy some property against any behavior of the environment
- How to automatically design a correct controller for the system?



Modelization

- Two-player zero-sum game played on a finite directed graph
- Property = objective for the system
- Synthesis of a controller = construction of a winning strategy



This talk

- General context
  - Focus on two-player zero-sum games for the synthesis of controllers
  - Extension to multiplayer non zero-sum games in the last part of the talk
  - Introductory survey with some classical results and some recent UMONS results

This talk

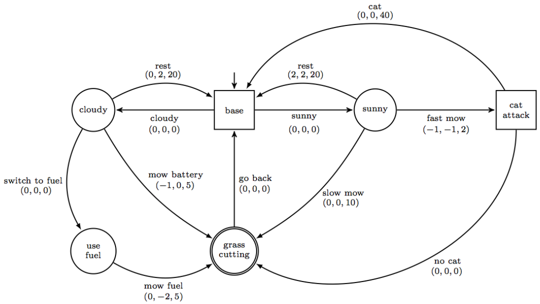
- General context
  - Focus on two-player zero-sum games for the synthesis of controllers
  - Extension to multiplayer non zero-sum games in the last part of the talk
  - Introductory survey with some classical results and some recent UMONS results
- Algorithmic problems
  - Does there exist a correct controller?
  - Can we construct it?
  - Is it possible to design a simple controller?

This talk

- General context
  - Focus on two-player zero-sum games for the synthesis of controllers
  - Extension to multiplayer non zero-sum games in the last part of the talk
  - Introductory survey with some classical results and some recent UMONS results
- Algorithmic problems
  - Does there exist a correct controller?
  - Can we construct it?
  - Is it possible to design a simple controller?
- More details
  - in my survey "Computer Aided Synthesis: a Game Theoretic Approach" in the Proceedings of DLT 2017 [Bru17]
  - in the book chapter "Graph Games and Reactive Synthesis" [BCJ18]
  - in the book chapter "Solution Concepts and Algorithms for Infinite Multiplayer Games" [GU08]

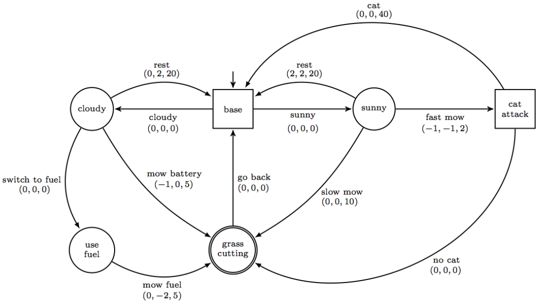
Introductory example

Lawnmower [Ran12]: modeled as a game played on a weighted graph



Introductory example

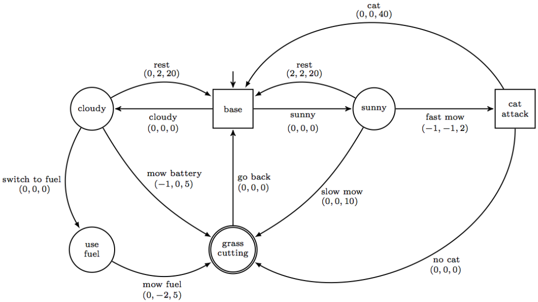
Lawnmower [Ran12]: modeled as a game played on a weighted graph



- Vertices: circles for the lawnmower, squares for the environment
- Edges: actions labeled by triples denoting changes in
  (solar battery, fuel level, elapsed time)

Introductory example

Lawnmower [Ran12]: modeled as a game played on a weighted graph

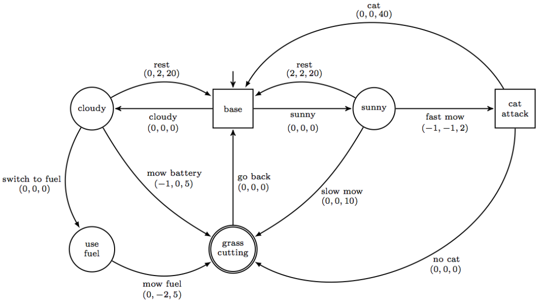


Specification as objectives

- Büchi objective : grass must be cut infinitely often
- Energy objective : battery and fuel must never drop below 0
- Mean-payoff objective : average time per action must be less than 10 in the long run

Introductory example

Lawnmower [Ran12]: modeled as a game played on a weighted graph



Controller as the following strategy

- If sunny, mow slowly
- If cloudy
  - If solar battery ≥ 1, mow on battery
  - otherwise, if fuel level ≥ 2, mow on fuel
  - otherwise, rest at the base

Basic model

Definition

Two-player zero-sum game $G = (V, V_1, V_2, E, v_0)$:

- (V, E) finite directed graph (with no deadlock)
- (V_1, V_2) partition of V with V_i controlled by player $i \in \{1, 2\}$
- initial vertex v_0

The players play in a turn-based way: they decide which edge (v, v') to follow for each v that they control

Basic model

Definition

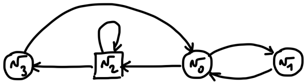
Two-player zero-sum game $G = (V, V_1, V_2, E, v_0)$:

- (V, E) finite directed graph (with no deadlock)
- (V_1, V_2) partition of V with V_i controlled by player $i \in \{1, 2\}$
- initial vertex v_0

The players play in a turn-based way: they decide which edge (v, v') to follow for each v that they control

Paths

- Play: infinite path from v_0 $\rho = \rho_0 \rho_1 \ldots \in V^\omega$ in G
- History: prefix h of a play



Player 1 ◯, Player 2 □

Basic model

Objective: set $\Omega \subseteq V^\omega$ of plays

Zero-sum game: objective Ω for player 1 and opposite objective $V^\omega \setminus \Omega$ for player 2

Basic model

Objective: set $\Omega \subseteq V^\omega$ of plays

Zero-sum game: objective Ω for player 1 and opposite objective $V^\omega \setminus \Omega$ for player 2

Definition

Some classical ω-regular objectives are:

- Reachability objective: visit a vertex of $U \subseteq V$ at least once
- Büchi objective: visit a vertex of U infinitely often
- Safety, Co-Büchi, Muller, Rabin, Streett

Basic model

Objective: set $\Omega \subseteq V^\omega$ of plays

Zero-sum game: objective Ω for player 1 and opposite objective $V^\omega \setminus \Omega$ for player 2

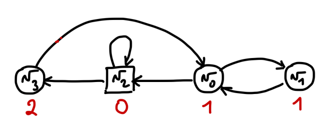
Definition

Some classical ω-regular objectives are:

- Reachability objective: visit a vertex of $U \subseteq V$ at least once
- Büchi objective: visit a vertex of U infinitely often
- Safety, Co-Büchi, Muller, Rabin, Streett

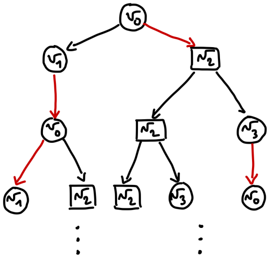
Given a coloring $c : V \to \mathbb{N}$

- Parity objective: the maximum color seen infinitely often is even



Strategies

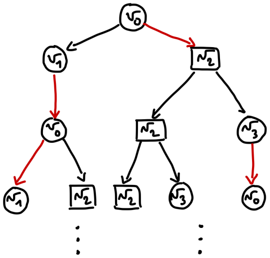
Strategy for player i:
function $\sigma_i : V^* V_i \to V$ such that
$\sigma_i(hv) = v'$ with $(v, v') \in E$



Unravelling of G from initial vertex v_0

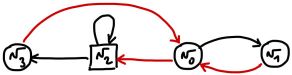
Strategies

Strategy for player i:
function $\sigma_i : V^* V_i \to V$ such that
$\sigma_i(hv) = v'$ with $(v, v') \in E$



Unravelling of G from initial vertex v_0

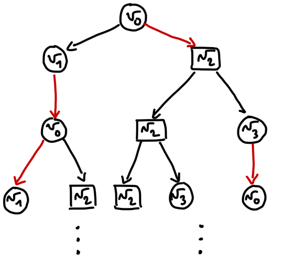
Positional strategy: when $\sigma_i(hv) = \sigma(v)$



Game G

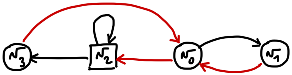
Strategies

Strategy for player i:
function $\sigma_i : V^* V_i \to V$ such that
$\sigma_i(hv) = v'$ with $(v, v') \in E$



Unravelling of G from initial vertex v_0

Positional strategy: when $\sigma_i(hv) = \sigma(v)$



Game G

Finite-memory strategy: when $\sigma_i(hv)$ only needs a finite information out of hv recorded in a finite-state machine

Strategies

Winning strategy for player i: ensure his objective against any strategy of the other player

A game is determined from initial vertex v_0 when

- either player 1 is winning for Ω from v_0
- or player 2 is winning for $V^\omega \setminus \Omega$ from v_0

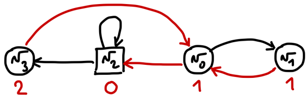
Strategies

Winning strategy for player i: ensure his objective against any strategy of the other player

A game is determined from initial vertex v_0 when

- either player 1 is winning for Ω from v_0
- or player 2 is winning for $V^\omega \setminus \Omega$ from v_0

Example



Parity game: Player 1 is winning from every vertex with a positional strategy

- Either player 2 eventually stays at v_2
  $\rightarrow$ max color seen infinitely often $= 0$
- Or he infinitely often visits v_3
  $\rightarrow$ max color seen infinitely often $= 2$

Martin's theorem

Theorem [Mar75]

Every game with Borel objectives is determined

Martin's theorem

Theorem [Mar75]

Every game with Borel objectives is determined

- Need of the axiom of choice to exhibit a non-determined game
- No information about the winning strategies

Martin's theorem

Theorem [Mar75]

Every game with Borel objectives is determined

- Need of the axiom of choice to exhibit a non-determined game
- No information about the winning strategies

Corollary

Every game with ω-regular objectives is determined

Martin's theorem

Theorem [Mar75]

Every game with Borel objectives is determined

- Need of the axiom of choice to exhibit a non-determined game
- No information about the winning strategies

Corollary

Every game with ω-regular objectives is determined

Algorithmic questions

- Who is the winner from initial vertex v_0?
- Complexity class of this decision problem?
- Can we construct a winning strategy for the winner?
- What kind of winning strategy? positional, finite-memory?

Algorithmic results for one-player games

Classical question in automata theory: Player 1 wins iff there exists a play satisfying the objective

- Reachability objective: emptiness of automata on finite words
- Büchi objective: emptiness of Büchi automata on infinite words

Algorithmic results for one-player games

Classical question in automata theory: Player 1 wins iff there exists a play satisfying the objective

- Reachability objective: emptiness of automata on finite words
- Büchi objective: emptiness of Büchi automata on infinite words

Finite-memory winning strategy iff the winning play is eventually periodic

Algorithmic results for one-player games

Classical question in automata theory: Player 1 wins iff there exists a play satisfying the objective

- Reachability objective: emptiness of automata on finite words
- Büchi objective: emptiness of Büchi automata on infinite words

Finite-memory winning strategy iff the winning play is eventually periodic

- reachable cycle in the graph
- reachable simple cycle for positional strategies

Algorithmic results for one-player games

Classical question in automata theory: Player 1 wins iff there exists a play satisfying the objective

- Reachability objective: emptiness of automata on finite words
- Büchi objective: emptiness of Büchi automata on infinite words

Finite-memory winning strategy iff the winning play is eventually periodic

- reachable cycle in the graph
- reachable simple cycle for positional strategies

Example

- Positional winning strategies for Reachability and Büchi objectives
- Needs of memory for Muller objective
  - visit all the vertices of $\{v_0, v_1, v_3\}$ infinitely often
  - Necessity to alternate

Algorithmic results for two-player games

Results [Bee80, EJ91, Imm81, Hor08], see also [GTW02, Zie98]

- Decision problem: who is the winner from initial vertex v_0?
- With what kind of winning strategy?

	Reach	Büchi	Parity	Muller
Complexity	P-complete		NP ∩ co-NP	P-complete
Player 1 strategy	positional			finite-memory
Player 2 strategy	positional			finite-memory

Algorithmic results for two-player games

Results [Bee80, EJ91, Imm81, Hor08], see also [GTW02, Zie98]

- Decision problem: who is the winner from initial vertex v_0?
- With what kind of winning strategy?

	Reach	Büchi	Parity	Muller
Complexity	P-complete		NP ∩ co-NP	P-complete
Player 1 strategy	positional			finite-memory
Player 2 strategy	positional			finite-memory

- Remember the previous examples
- More information on the proofs on slide 21

Algorithmic results for two-player games

Results [Bee80, EJ91, Imm81, Hor08], see also [GTW02, Zie98]

- Decision problem: who is the winner from initial vertex v_0?
- With what kind of winning strategy?

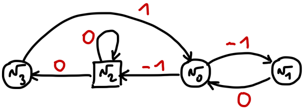
	Reach	Büchi	Parity	Muller
Complexity	P-complete		NP ∩ co-NP	P-complete
Player 1 strategy	positional			finite-memory
Player 2 strategy	positional			finite-memory

- Remember the previous examples
- More information on the proofs on slide 21

Major open problem: can we solve Parity games in P?
Recent breakthrough with a quasi-polynomial time algorithm [CJK+17]

Basic model

Extension with weights on the edges



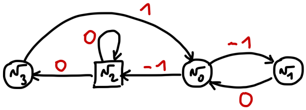
Definition

Two-player zero-sum game $G = (V, V_1, V_2, E, v_0, w)$ as before, with:

- $w : E \to \mathbb{Z}$ weight function

Basic model

Extension with weights on the edges



Definition

Two-player zero-sum game $G = (V, V_1, V_2, E, v_0, w)$ as before, with:

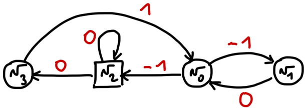
- $w : E \to \mathbb{Z}$ weight function

Classical payoff $f(\rho)$ of a play $\rho = \rho_0\rho_1\rho_2\ldots$:

- $\mathrm{Sup}(\rho) = \sup_{n\in\mathbb{N}} w(\rho_n, \rho_{n+1})$
- $\mathrm{LimSup}(\rho) = \limsup_{n\to\infty} w(\rho_n, \rho_{n+1})$
- Mean-payoff $\overline{\mathrm{MP}}(\rho) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} w(\rho_k, \rho_{k+1})$
- Discounted sum $\mathrm{Disc}^{\lambda}(\rho) = \sum_{k=0}^{\infty} w(\rho_k, \rho_{k+1})\lambda^n$, where $\lambda \in]0,1[$

Basic model

Extension with weights on the edges



Definition

Two-player zero-sum game $G = (V, V_1, V_2, E, v_0, w)$ as before, with:

- $w : E \to \mathbb{Z}$ weight function

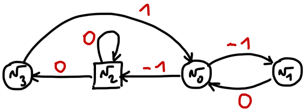
Classical payoff $f(\rho)$ of a play $\rho = \rho_0\rho_1\rho_2 \ldots$:

- Sup(ρ) $= \sup_{n\in\mathbb{N}} w(\rho_n, \rho_{n+1})$
- LimSup(ρ) $= \limsup_{n\to\infty} w(\rho_n, \rho_{n+1})$
- Mean-payoff $\overline{\text{MP}}(\rho) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} w(\rho_k, \rho_{k+1})$
- Discounted sum $\text{Disc}^{\lambda}(\rho) = \sum_{k=0}^{\infty} w(\rho_k, \rho_{k+1})\lambda^n$, where $\lambda \in]0, 1[$

Similar definitions with $\text{Inf}(\rho), \text{LimInf}(\rho), \underline{\text{MP}}(\rho)$

Quantitative objectives

Example



play $\rho = (v_0 v_1)^\omega$

- $\overline{\mathrm{MP}}(\rho) = \limsup\limits_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} w(\rho_k, \rho_{k+1})$

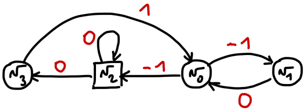
  $(\frac{-1}{1}, \frac{-1}{2}, \frac{-2}{3}, \frac{-2}{4}, \frac{-3}{5}, \frac{-3}{6}, \ldots, \frac{-n}{2n-1}, \frac{-n}{2n}, \ldots) \to -\frac{1}{2} = \overline{\mathrm{MP}}(\rho) = \underline{\mathrm{MP}}(\rho)$

- $\mathrm{Disc}^\lambda(\rho) = \sum_{k=0}^{\infty} w(\rho_k, \rho_{k+1}) \lambda^n$, with $\lambda = \frac{1}{2}$

  $-1 - \lambda^2 - \lambda^4 - \lambda^6 \ldots = -\frac{4}{3}$

Quantitative objectives

Example



play $\rho = (v_0 v_1)^\omega$

- $\overline{\text{MP}}(\rho) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} w(\rho_k, \rho_{k+1})$

  $(\frac{-1}{1}, \frac{-1}{2}, \frac{-2}{3}, \frac{-2}{4}, \frac{-3}{5}, \frac{-3}{6}, \ldots, \frac{-n}{2n-1}, \frac{-n}{2n}, \ldots) \to -\frac{1}{2} = \overline{\text{MP}}(\rho) = \underline{\text{MP}}(\rho)$

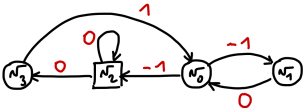
- $\text{Disc}^\lambda(\rho) = \sum_{k=0}^{\infty} w(\rho_k, \rho_{k+1})\lambda^n$, with $\lambda = \frac{1}{2}$

  $-1 - \lambda^2 - \lambda^4 - \lambda^6 \ldots = -\frac{4}{3}$

Lemma: If $\rho = hg^\omega$ is eventually periodic, then $\overline{\text{MP}}(\rho) = \underline{\text{MP}}(\rho) =$ average weight of cycle g

Quantitative objectives

Example



play $\rho = (v_0 v_1)^\omega$

- $\overline{\mathrm{MP}}(\rho) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} w(\rho_k, \rho_{k+1})$
  $(\frac{-1}{1}, \frac{-1}{2}, \frac{-2}{3}, \frac{-2}{4}, \frac{-3}{5}, \frac{-3}{6}, \ldots, \frac{-n}{2n-1}, \frac{-n}{2n}, \ldots) \to -\frac{1}{2} = \overline{\mathrm{MP}}(\rho) = \underline{\mathrm{MP}}(\rho)$
- $\mathrm{Disc}^\lambda(\rho) = \sum_{k=0}^{\infty} w(\rho_k, \rho_{k+1})\lambda^n$, with $\lambda = \frac{1}{2}$
  $-1 - \lambda^2 - \lambda^4 - \lambda^6 \ldots = -\frac{4}{3}$

Lemma: If $\rho = hg^\omega$ is eventually periodic, then $\overline{\mathrm{MP}}(\rho) = \underline{\mathrm{MP}}(\rho) =$ average weight of cycle g

Remark: For qualitative objectives, Boolean payoff $f(\rho) \in \{0, 1\}$

Quantitative objectives

Definition

Classical quantitative objectives are:

- Threshold problem: given a threshold $\mu \in \mathbb{Q}$:
  - Sup objective: ensure $\mu \leq \mathsf{Sup}(\rho)$
  - Similarly for the other payoff functions LimSup, $\overline{\mathsf{MP}}$, ...
- Constraint problem: given a rational interval $[\mu, \nu]$,
  - ensure $\mu \leq \mathsf{Sup}(\rho) \leq \nu$

Quantitative objectives

Definition

Classical quantitative objectives are:

- Threshold problem: given a threshold $\mu \in \mathbb{Q}$:
  - Sup objective: ensure $\mu \leq \mathsf{Sup}(\rho)$
  - Similarly for the other payoff functions LimSup, $\overline{\mathsf{MP}}$, ...
- Constraint problem: given a rational interval $[\mu, \nu]$,
  - ensure $\mu \leq \mathsf{Sup}(\rho) \leq \nu$

Corollary of Martin's Theorem

Games with such quantitative objectives are determined

- Sup, LimSup: ω-regular
- $\overline{\mathsf{MP}}$, Disc^{λ}: not ω-regular, but Borel

Quantitative objectives

Definition

Classical quantitative objectives are:

- Threshold problem: given a threshold $\mu \in \mathbb{Q}$:
  - Sup objective: ensure $\mu \leq \mathsf{Sup}(\rho)$
  - Similarly for the other payoff functions LimSup, $\overline{\mathsf{MP}}$, ...
- Constraint problem: given a rational interval $[\mu, \nu]$,
  - ensure $\mu \leq \mathsf{Sup}(\rho) \leq \nu$

Corollary of Martin's Theorem

Games with such quantitative objectives are determined

- ensure $\mu \leq \mathsf{Sup}(\rho) \Leftrightarrow$ visit an edge with a weight $\geq \mu$ (Reachability)
- ensure $\mu \leq \mathsf{LimSup}(\rho) \Leftrightarrow$ visit such an edge infinitely often (Büchi)

Algorithmic results for one-player games

Theorem [CDH10]

Polynomial time algorithm for the threshold problem, with positional winning strategies

More information on the proofs on slide 21

Algorithmic results for one-player games

Theorem [CDH10]

Polynomial time algorithm for the threshold problem, with positional winning strategies

More information on the proofs on slide 21

Theorem [HR14, UW11a]

For the constraint problem, polynomial time algorithm

- Sup, Inf, LimSup, LimInf: positional winning strategies
- $\overline{\mathsf{MP}}$, $\underline{\mathsf{MP}}$: finite-memory winning strategies

Open for Disc^{λ}

Algorithmic results for one-player games

Theorem [CDH10]

Polynomial time algorithm for the threshold problem, with positional winning strategies

More information on the proofs on slide 21

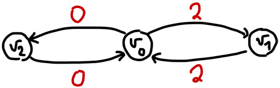
Theorem [HR14, UW11a]

For the constraint problem, polynomial time algorithm

- Sup, Inf, LimSup, LimInf: positional winning strategies
- $\overline{\text{MP}}$, $\underline{\text{MP}}$: finite-memory winning strategies

Open for Disc^{λ}

Example
Mean-payoff with $\mu = \nu = 1$
Necessity to alternate



Algorithmic results for one-player games

Two related open problems:

Constraint problem for Disc^λ

Given a rational interval $[\mu, \nu]$, does there exist a play ρ such that $\mu \leq \mathsf{Disc}^\lambda(\rho) \leq \nu$?

Target discounted-sum (TDS) problem [BHO15]

Given four rational numbers a, b, t and $\lambda \in]0, 1[$, does there exist an infinite sequence $u = u_0 u_1 \ldots \in \{a, b\}^\omega$ such that $\sum_{n=0}^{\infty} u_n \lambda^n = t$?

Algorithmic results for one-player games

Two related open problems:

Constraint problem for Disc^λ

Given a rational interval $[\mu, \nu]$, does there exist a play ρ such that $\mu \leq \mathsf{Disc}^\lambda(\rho) \leq \nu$?

Target discounted-sum (TDS) problem [BHO15]

Given four rational numbers a, b, t and $\lambda \in]0,1[$, does there exist an infinite sequence $u = u_0 u_1 \ldots \in \{a, b\}^\omega$ such that $\sum_{n=0}^{\infty} u_n \lambda^n = t$?

- TDS problem related to several open questions in mathematics and computer science [BHO15]
- related to numeration systems, i.e. to β-representations of real numbers [Ŕ57]
- TDS problem decidable when $a = 0, b = 1$ and $\lambda \geq \frac{1}{2}$ [Ŕ57]

Algorithmic results for two-player games

Threshold problem [BSV04, EM79, ZP96]

	Reach Sup	Büchi LimSup	Parity	$\overline{\mathsf{MP}}$	Disc^{λ}	Muller
Complexity	P-complete		NP $\cap$ co-NP			P-complete
Player 1 strategy	positional					finite-memory
Player 2 strategy	positional					finite-memory

Algorithmic results for two-player games

Threshold problem [BSV04, EM79, ZP96]

	Reach Sup	Büchi LimSup	Parity	$\overline{\text{MP}}$	Disc^λ	Muller
Complexity	P-complete		NP $\cap$ co-NP			P-complete
Player 1 strategy	positional					finite-memory
Player 2 strategy	positional					finite-memory

- Polynomial reductions: Parity games $\rightarrow$ Mean-payoff games $\rightarrow$ Discounted-sum games [Jur98, ZP96]
- Major open problem: can we solve Parity, Mean-payoff and Discounted-sum games in P?

Algorithmic results for two-player games

Threshold problem [BSV04, EM79, ZP96]

	Reach Sup	Büchi LimSup	Parity	$\overline{\text{MP}}$	Disc^λ	Muller
Complexity	P-complete		NP ∩ co-NP			P-complete
Player 1 strategy	positional					finite-memory
Player 2 strategy	positional					finite-memory

- Polynomial reductions: Parity games → Mean-payoff games → Discounted-sum games [Jur98, ZP96]
- Major open problem: can we solve Parity, Mean-payoff and Discounted-sum games in P?

Constraint problem [HR14, UW11a] Same results except

- finite-memory strategies for $\overline{\text{MP}}$
- open for Disc^λ

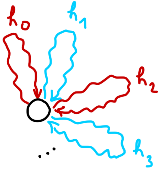
Algorithmic results

Theorem [GZ04]

Let G be a weighted game. If the payoff function f is fairly mixing, i.e.:

1. $f(\rho) \leq f(\rho') \Rightarrow f(h\rho) \leq f(h\rho')$
2. $\min\{f(\rho), f(h^\omega)\} \leq f(h\rho) \leq \max\{f(\rho), f(h^\omega)\}$
3. $\min\{f(h_0 h_2 h_4 \ldots), f(h_1 h_3 h_5 \ldots), \inf_i f(h_i^\omega)\}$ $\leq f(h_0 h_1 h_2 h_3 \ldots) \leq \max\{f(h_0 h_2 h_4 \ldots), f(h_1 h_3 h_5 \ldots), \sup_i f(h_i^\omega)\}$

then both players have positional winning strategies for the threshold problem



Algorithmic results

Theorem [GZ04]

Let G be a weighted game. If the payoff function f is fairly mixing, i.e.:

1. $f(\rho) \leq f(\rho') \Rightarrow f(h\rho) \leq f(h\rho')$
2. $\min\{f(\rho), f(h^\omega)\} \leq f(h\rho) \leq \max\{f(\rho), f(h^\omega)\}$
3. $\min\{f(h_0h_2h_4\ldots), f(h_1h_3h_5\ldots), \inf_i f(h_i^\omega)\}$
   $\leq f(h_0h_1h_2h_3\ldots) \leq \max\{f(h_0h_2h_4\ldots), f(h_1h_3h_5\ldots), \sup_i f(h_i^\omega)\}$

then both players have positional winning strategies for the threshold problem

- Many applications: Reachability, Büchi, Parity, Sup, LimSup, Mean-payoff, Discounted-sum, ... (but not Muller)
- If the payoff function is prefix-independent, i.e. $f(\rho) = f(h\rho)$, then conditions 1. and 2. are satisfied
- Simple proof by induction on the number of edges

Algorithmic results

Parity games in NP $\cap$ co-NP

Algorithmic results

Parity games in NP ∩ co-NP

- in NP:
  - Guess a positional winning strategy σ player 1
  - Construct the one-player game G_σ obtained from G by fixing σ
  - Check in polynomial time whether there exists a reachable cycle with odd maximum color

Algorithmic results

Parity games in NP $\cap$ co-NP

- in NP:
  - Guess a positional winning strategy σ player 1
  - Construct the one-player game G_σ obtained from G by fixing σ
  - Check in polynomial time whether there exists a reachable cycle with odd maximum color
- in co-NP: symmetrically for player 2

Algorithmic results

Parity games in NP ∩ co-NP

- in NP:
  - Guess a positional winning strategy σ player 1
  - Construct the one-player game G_σ obtained from G by fixing σ
  - Check in polynomial time whether there exists a reachable cycle with odd maximum color
- in co-NP: symmetrically for player 2

Mean-payoff games in NP ∩ co-NP

Algorithmic results

Parity games in NP $\cap$ co-NP

- in NP:
  - Guess a positional winning strategy σ player 1
  - Construct the one-player game G_σ obtained from G by fixing σ
  - Check in polynomial time whether there exists a reachable cycle with odd maximum color
- in co-NP: symmetrically for player 2

Mean-payoff games in NP $\cap$ co-NP

- Same approach
- One can compute in polynomial time the minimum (resp. maximum) average weight cycle in a weighted graph [Kar78]

Intersection of objectives

- Intersection of homogeneous objectives
  - For instance, intersection of n reachability objectives

Intersection of objectives

- Intersection of homogeneous objectives
  - For instance, intersection of *n* reachability objectives
- Intersection of heterogeneous objectives
  - Remember the lawnmower example
    - Büchi objective : grass must be cut infinitely often
    - Energy objective : battery and fuel must never drop below 0
    - Mean-payoff objective : average time per action must be less than 10 in the long run

Intersection of objectives

- Intersection of homogeneous objectives
  - For instance, intersection of n reachability objectives
- Intersection of heterogeneous objectives
  - Remember the lawnmower example
    - Büchi objective : grass must be cut infinitely often
    - Energy objective : battery and fuel must never drop below 0
    - Mean-payoff objective : average time per action must be less than 10 in the long run

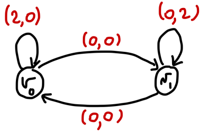
Orderings on tuples of payoffs

- Usual (partial) ordering (see next slides)
  $(x_1, y_1) \geq_{comp} (x_2, y_2)$ iff $x_1 \geq x_2$ and $y_1 \geq y_2$
- Lexicographic ordering [BBMU15], [BHR17]
  $(x_1, y_1) \geq_{lex} (x_2, y_2)$ iff $x_1 > x_2$ or $(x_1 = x_2$ and $y_1 \geq y_2)$
- Orderings given by Boolean circuits [BBMU15]

Intersection of objectives

Example

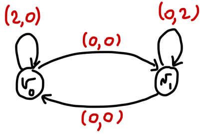
- One-player game, order $\geq_{comp}$
- $k = 2$, $\Omega = \Omega_1 \cap \Omega_2$
  with $\Omega_1 = \overline{\text{MP}}(\rho) \geq 1$ for dimension 1
  and $\Omega_2 = \overline{\text{MP}}(\rho) \geq 1$ for dimension 2



Intersection of objectives

Example

- One-player game, order $\geq_{comp}$
- $k = 2$, $\Omega = \Omega_1 \cap \Omega_2$
  with $\Omega_1 = \overline{\text{MP}}(\rho) \geq 1$ for dimension 1
  and $\Omega_2 = \overline{\text{MP}}(\rho) \geq 1$ for dimension 2



- Player 1 is losing with finite-memory strategies
  - Eventually periodic play $\rho = hg^\omega$
  - Average weight of cycle g equal to

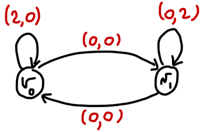
$$a \cdot (2,0) + b \cdot (0,0) + c \cdot (0,2) = (2 \cdot a, 2 \cdot c) \not\geq_{comp} (1,1)$$

with $a + b + c = 1$ and $b > 0$

Intersection of objectives

Example

- One-player game, order $\geq_{comp}$
- $k = 2$, $\Omega = \Omega_1 \cap \Omega_2$
  with $\Omega_1 = \overline{\mathrm{MP}}(\rho) \geq 1$ for dimension 1
  and $\Omega_2 = \overline{\mathrm{MP}}(\rho) \geq 1$ for dimension 2



- Player 1 is losing with finite-memory strategies
  - Eventually periodic play $\rho = hg^\omega$
  - Average weight of cycle g equal to

    $$a \cdot (2,0) + b \cdot (0,0) + c \cdot (0,2) = (2 \cdot a, 2 \cdot c) \not\geq_{comp} (1,1)$$

    with $a + b + c = 1$ and $b > 0$

- Player 1 is winning with infinite-memory strategies,
  - even for ensuring $\geq_{comp} (2,2)$
  - with $\underline{\mathrm{MP}}$ instead of $\overline{\mathrm{MP}}$, but only for ensuring $\geq_{comp} (1,1)$

Intersection of objectives

Homogeneous objectives [CDHR10, CHP07, FH13],[CRR14]

	Reach	Parity	$\underline{\text{MP}}$	$\overline{\text{MP}}$
Complexity	PSPACE-complete	coNP-complete		NP ∩ co-NP
Pl. 1 strategy	finite-memory		infinite-memory	
Pl. 2 strategy	finite-memory	positional		

Intersection of objectives

Homogeneous objectives [CDHR10, CHP07, FH13],[CRR14]

	Reach	Parity	$\underline{\text{MP}}$	$\overline{\text{MP}}$
Complexity	PSPACE-complete	coNP-complete		NP ∩ co-NP
Pl. 1 strategy	finite-memory		infinite-memory	
Pl. 2 strategy	finite-memory	positional		

Heterogeneous objectives

Theorem

- [Vel15]: Undecidability for Boolean comb. of $\underline{\text{MP}}$ and $\overline{\text{MP}}$ objectives
- [BHR16]: PSPACE-completeness for Boolean combinations of Inf, Sup, LimInf, LimSup objectives, and finite-memory winning strategies for both players

- [BHRR19]: Work in progress about the intersection of two objectives: mean-payoff and energy

Another model

Summary

- Reactive systems embedded into an uncontrollable environment
  - 2-player zero-sum games, one player against the other
  - Qualitative/quantitative uni/multidimensional objectives

Another model

Summary

- Reactive systems embedded into an uncontrollable environment
  - 2-player zero-sum games, one player against the other
  - Qualitative/quantitative uni/multidimensional objectives

Another model

- Systems composed of multiple interacting components
  - Each component (= player) has objectives that can be compatible or not with the objectives of the other components

Another model

Summary

- Reactive systems embedded into an uncontrollable environment
  - 2-player zero-sum games, one player against the other
  - Qualitative/quantitative uni/multidimensional objectives

Another model

- Systems composed of multiple interacting components
  - Each component (= player) has objectives that can be compatible or not with the objectives of the other components
- Modelization with multiplayer non zero-sum games played on graphs
  - Several players with their own objectives
  - Non necessarily antagonistic objectives
  - Focus on the synthesis of equilibria instead of winning srategies

Another model

Summary

- Reactive systems embedded into an uncontrollable environment
  - 2-player zero-sum games, one player against the other
  - Qualitative/quantitative uni/multidimensional objectives

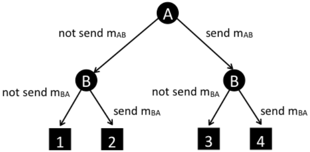
Another model

- Systems composed of multiple interacting components
  - Each component (= player) has objectives that can be compatible or not with the objectives of the other components
- Modelization with multiplayer non zero-sum games played on graphs
  - Several players with their own objectives
  - Non necessarily antagonistic objectives
  - Focus on the synthesis of equilibria instead of winning srategies
- Algorithmic problems
  - Does there always exist an equilibrium? Can we construct it?
  - Can we decide the existence of an equilibrium under some constraints?

Introductory example

Exchange protocol [CDFR17]

- Players Alice (A) and Bob (B) exchange messages
  - message m_{AB}: models the transfer of property of a house from A to B
  - message m_{BA}: models the payment of the price of the house from B to A

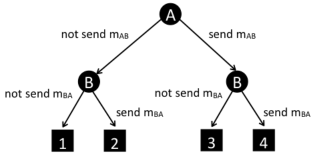


Tree with 4 plays

Introductory example

Exchange protocol [CDFR17]

- Players Alice (A) and Bob (B) exchange messages
  - message m_{AB}: models the transfer of property of a house from A to B
  - message m_{BA}: models the payment of the price of the house from B to A



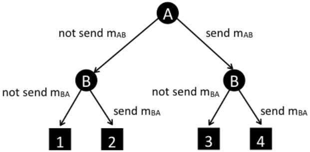
Tree with 4 plays

- Alice and Bob have their own objectives
  - Objective of A: to get the money (she prefers $2, 4$ to $1, 3$)
  - Objective of B: to get the house (he prefers $3, 4$ to $1, 2$)

Introductory example

Exchange protocol [CDFR17]

- Players Alice (A) and Bob (B) exchange messages
  - message m_{AB}: models the transfer of property of a house from A to B
  - message m_{BA}: models the payment of the price of the house from B to A



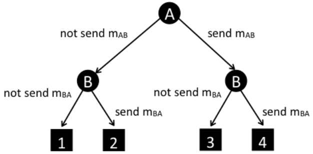
Tree with 4 plays

- Alice and Bob have their own objectives
  - Objective of A: to get the money (she prefers $2, 4$ to $1, 3$)
  - Objective of B: to get the house (he prefers $3, 4$ to $1, 2$)
- Solution (Nash equilibrium)
  - A plays m_{AB} and then B plays m_{BA}
  - if A plays $\neg m_{AB}$, then B plays $\neg m_{BA}$
- A and B have their objective satisfied and have no incentive to deviate

Introductory example

Modified exchange protocol [CDFR17]

- Alice and Bob have their own primary objective and external secondary objective
  - Objectives of A:
    1. prefers $2, 4$ to $1, 3$
    2. with preferences $2 > 4$ and $1 > 3$
  - Objectives of B:
    1. prefers $3, 4$ to $1, 2$
    2. with preferences $3 > 4$ and $1 > 2$

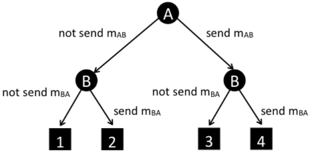


Tree with 4 plays

Introductory example

Modified exchange protocol [CDFR17]

- Alice and Bob have their own primary objective and external secondary objective
  - Objectives of A:
    1. prefers $2, 4$ to $1, 3$
    2. with preferences $2 > 4$ and $1 > 3$
  - Objectives of B:
    1. prefers $3, 4$ to $1, 2$
    2. with preferences $3 > 4$ and $1 > 2$



Tree with 4 plays

- Solution
  - Play 4 is no longer a solution since B will deviate to play 3
  - Solution (secure equilibrium):
    - A plays $\neg m_{AB}$ and then B plays $\neg m_{BA}$
    - If A plays m_{AB}, B plays $\neg m_{BA}$
  - A, B have no incentive to deviate, their own objectives are not satisfied

Model

Definition

n-player non zero-sum game $G = (V, (V_i)_{i \in \Pi}, E, v_0, \bar{w})$:

- Set Π of n players, $n \geq 1$
- $(V_i)_{i \in \Pi}$ partition of V with V_i controlled by player $i \in \Pi$
- optional: $\bar{w} = (w_i)_{i \in \Pi} : E \to \mathbb{Z}^n$ such that
  - w_i is the weight function of player i
  - leading to his payoff function f_i

Model

Definition

n-player non zero-sum game $G = (V, (V_i)_{i \in \Pi}, E, v_0, \bar{w})$:

- Set Π of n players, $n \geq 1$
- $(V_i)_{i \in \Pi}$ partition of V with V_i controlled by player $i \in \Pi$
- optional: $\bar{w} = (w_i)_{i \in \Pi} : E \to \mathbb{Z}^n$ such that
  - w_i is the weight function of player i
  - leading to his payoff function f_i

Objective Ω_i for each player $i \in \Pi$

- qualitative ($f_i(\rho) \in \{0, 1\}$): player i wants to win
- quantitative : player i wants to maximize $f_i(\rho)$

Model

Definition

n-player non zero-sum game $G = (V, (V_i)_{i \in \Pi}, E, v_0, \bar{w})$:

- Set Π of n players, $n \geq 1$
- $(V_i)_{i \in \Pi}$ partition of V with V_i controlled by player $i \in \Pi$
- optional: $\bar{w} = (w_i)_{i \in \Pi} : E \to \mathbb{Z}^n$ such that
  - w_i is the weight function of player i
  - leading to his payoff function f_i

Objective Ω_i for each player $i \in \Pi$

- qualitative ($f_i(\rho) \in \{0, 1\}$): player i wants to win
- quantitative : player i wants to maximize $f_i(\rho)$

Strategy profile $(\sigma_i)_{i \in \Pi}$

- with outcome $\rho = \langle (\sigma_i)_{i \in \Pi} \rangle_{v_0}$ from initial vertex v_0
- with payoff $(f_i(\rho))_{i \in \Pi}$

Nash equilibria

Classical notion such that

- each player wants to maximize his payoff (he is rational), and
- he is only concerned with his own payoff (he is indifferent to the payoff of the other players)

Nash equilibria

Classical notion such that

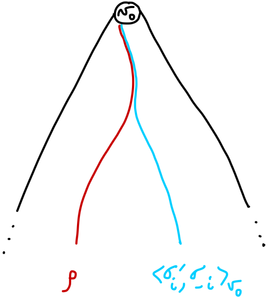
- each player wants to maximize his payoff (he is rational), and
- he is only concerned with his own payoff (he is indifferent to the payoff of the other players)

Definition [Nas50]

The strategy profile $(\sigma_i)_{i \in \Pi}$ with outcome ρ from v_0 is a Nash equilibrium (NE) if, for each player $i \in \Pi$, for each strategy σ'_i of i,

$$f_i(\rho) \not< f_i(\langle \sigma'_i, \sigma_{-i} \rangle_{v_0})$$

Notation: $\sigma_{-i} = (\sigma_j)_{j \in \Pi \setminus \{i\}}$



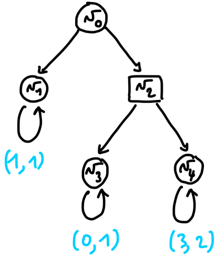
Informally, $(\sigma_i)_{i \in \Pi}$ is an NE if no player has an incentive to deviate from his strategy, if the other players stick to their own strategies

Nash equilibria

Example

Simple game

- with 2 players
- with 3 plays
- and their payoffs indicated below

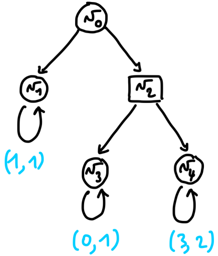


Nash equilibria

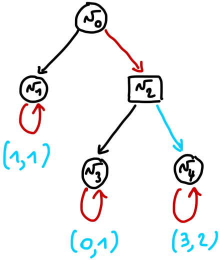
Example

Simple game

- with 2 players
- with 3 plays
- and their payoffs indicated below



- NE with outcome $v_0 v_2 v_4^\omega$ with payoff $(3, 2)$
- No incentive to deviate:
  - If player 1 deviates to v_1, he will get 1 instead of 3
  - If player 2 deviates to v_3, he will get 1 instead of 2



Algorithmic results on NE

Theorem

Qualitative objectives

- [GU08]: Existence of an NE in case of Borel objectives

Quantitative objectives

- [Kuh53]: Existence and construction of an NE for games played on a finite tree

Algorithmic results on NE

Theorem

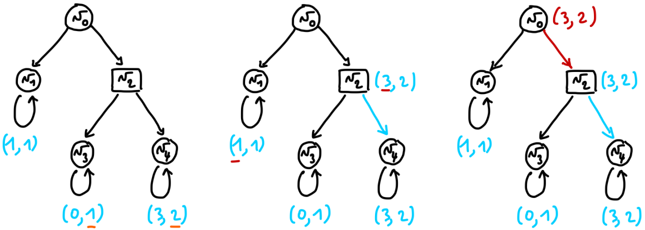
Qualitative objectives

- [GU08]: Existence of an NE in case of Borel objectives

Quantitative objectives

- [Kuh53]: Existence and construction of an NE for games played on a finite tree

Proof of [Kuh53]: Backward induction from the leaves to the root



Algorithmic results on NE

Definition

Given a game G and a player i,

- G_i is a two-player zero-sum game with players i and $-i$ (coalition), and payoff function f_i
- In G_i, a vertex v has a value $val_i(v)$ if
  - player i has a strategy τ_i^v to ensure a payoff $\geq \mathrm{val}_i(v)$ from v
  - player $-i$ has a strategy τ_{-i}^v to ensure a payoff $\leq \mathrm{val}_i(v)$ from v
- The strategies τ_i^v and τ_{-i}^v are called optimal

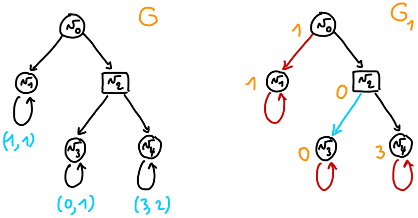
Algorithmic results on NE

Definition

Given a game G and a player i,

- G_i is a two-player zero-sum game with players i and $-i$ (coalition), and payoff function f_i
- In G_i, a vertex v has a value $val_i(v)$ if
  - player i has a strategy τ_i^v to ensure a payoff $\geq \text{val}_i(v)$ from v
  - player $-i$ has a strategy τ_{-i}^v to ensure a payoff $\leq \text{val}_i(v)$ from v
- The strategies τ_i^v and τ_{-i}^v are called optimal

Example



Algorithmic results on NE

Theorem [BDS13]

Let G be a multiplayer non zero-sum game such that for all i

- the payoff function f_i satisfies: $f_i(\rho) \leq f_i(\rho') \Rightarrow f_i(h\rho) \leq f_i(h\rho')$
- the zero-sum game G_i has uniform positional optimal τ_i and τ_{-i} strategies for both players

Then one can construct a simple finite-memory NE in G

Algorithmic results on NE

Theorem [BDS13]

Let G be a multiplayer non zero-sum game such that for all i

- the payoff function f_i satisfies: $f_i(\rho) \leq f_i(\rho') \Rightarrow f_i(h\rho) \leq f_i(h\rho')$
- the zero-sum game G_i has uniform positional optimal τ_i and τ_{-i} strategies for both players

Then one can construct a simple finite-memory NE in G

Construction: The NE profile σ is as follows

- play as τ_i for each player i
  (player i plays selfishly and optimally with respect to f_i)
- and as soon as some player i deviates, punish i by playing τ_{-i}
  (coalition $-i$ plays against player i with respect to f_i)

Algorithmic results on NE

Theorem [BDS13]

Let G be a multiplayer non zero-sum game such that for all i

- the payoff function f_i satisfies: $f_i(\rho) \leq f_i(\rho') \Rightarrow f_i(h\rho) \leq f_i(h\rho')$
- the zero-sum game G_i has uniform positional optimal τ_i and τ_{-i} strategies for both players

Then one can construct a simple finite-memory NE in G

Many applications

- Reachability, Büchi, Parity, Sup, LimSup, Mean-payoff, Discounted-sum, ...

Algorithmic results on NE

Theorem [BDS13]

Let G be a multiplayer non zero-sum game such that for all i

- the payoff function f_i satisfies: $f_i(\rho) \leq f_i(\rho') \Rightarrow f_i(h\rho) \leq f_i(h\rho')$
- the zero-sum game G_i has uniform positional optimal τ_i and τ_{-i} strategies for both players

Then one can construct a simple finite-memory NE in G

Threshold/constraint problem for NEs [Umm08, UW11b, KLST12]

Büchi/LimSup	Reach/Sup	Parity	$\overline{\text{MP}}$
P-complete	NP-complete		

- thanks to a characterization of NE outcomes based on games G_i
- Open for Disc^λ

Other kinds of equilibria

Secure equilibrium (SE) [CHJ06]

- each player wants to maximize his payoff, as a first objective
- and then minimize the payoff of the other players, as a second objective

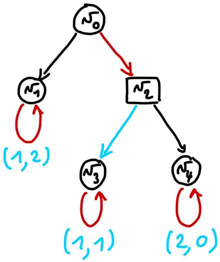
Other kinds of equilibria

Secure equilibrium (SE) [CHJ06]

- each player wants to maximize his payoff, as a first objective
- and then minimize the payoff of the other players, as a second objective

Example

- SE with payoff $(1, 1)$
- No incentive to deviate
  - If Player 2 deviates, he gets 0 instead of 1
  - If Player 1 deviates, he keeps his payoff 1 but he increases the payoff of Player 2



Other kinds of equilibria

Secure equilibrium (SE) [CHJ06]

- each player wants to maximize his payoff, as a first objective
- and then minimize the payoff of the other players, as a second objective

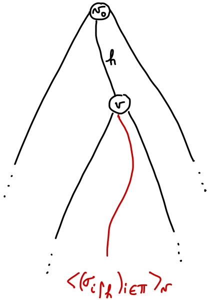
Theorem

- [CHJ06]: Existence of an SE for 2-player games with qualitative Borel objectives. Result extended to n-player games in [DFK$^+$14]
- [BMR14]: Previous general approach for NEs extended to SE for 2-player games

Other kinds of equilibria

Subgame perfect equilibrium (SPE) [Sel65]

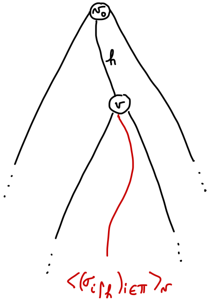
- takes into account the sequential nature of games played on graphs
- i.e., is an NE from the initial vertex v_0, but also after every history *h* of the game
- avoids uncredible threat



Other kinds of equilibria

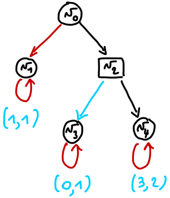
Subgame perfect equilibrium (SPE) [Sel65]

- takes into account the sequential nature of games played on graphs
- i.e., is an NE from the initial vertex v_0, but also after every history h of the game
- avoids uncredible threat



Example: NE which is not an SPE

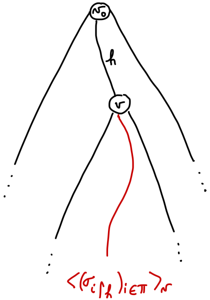
- Player 1 will not deviate, due to the threat of player 2
- Uncredible threat of player 2
- More rational for player 2 to go to v_4 in the subgame induced by v_2, v_3, v_4



Other kinds of equilibria

Subgame perfect equilibrium (SPE) [Sel65]

- takes into account the sequential nature of games played on graphs
- i.e., is an NE from the initial vertex v_0, but also after every history h of the game
- avoids uncredible threat



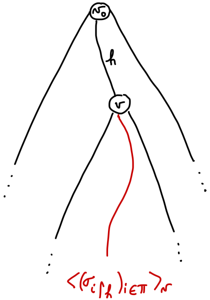
Theorem

- Previous result of [Kuh53] provides NE and more generally SPE
- [GU08]: Existence of an SPE in case of qualitative Borel objectives
- [SV03]: Simple example of a game with mean-payoff objectives that has no SPE

Other kinds of equilibria

Subgame perfect equilibrium (SPE) [Sel65]

- takes into account the sequential nature of games played on graphs
- i.e., is an NE from the initial vertex v_0, but also after every history h of the game
- avoids uncredible threat



Theorem

- [BBGR18]: The constraint problem for SPE with reachability objectives is PSPACE-complete
- [BBMR15]: Construction of a finite-memory SPE for quantitative reachability

- [BBG+19] (Work in progress): The constraint problem for SPE in quantitative reachability games is PSPACE-complete

Summary

- 2-player zero-sum games, one player against the other
- Qualitative/quantitative uni/multidimensional objectives
- Extension to multiplayer non zero-sum games
- Different notions of equilibria (NE, SE, SPE)
- More results in the survey

Summary

- 2-player zero-sum games, one player against the other
- Qualitative/quantitative uni/multidimensional objectives
- Extension to multiplayer non zero-sum games
- Different notions of equilibria (NE, SE, SPE)
- More results in the survey

Other extensions

- Concurrent games
- Stochastic games
- Imperfect information

Summary

- 2-player zero-sum games, one player against the other
- Qualitative/quantitative uni/multidimensional objectives
- Extension to multiplayer non zero-sum games
- Different notions of equilibria (NE, SE, SPE)
- More results in the survey

Other extensions

- Concurrent games
- Stochastic games
- Imperfect information

Thank you!

Thomas Brihaye, Véronique Bruyère, Aline Goeminne, Jean-François Raskin, and Marie Van Den Bogaard, *Constrained problem for subgame perfect equilibria in quantitative reachability games*, Work in progress, 2019.

Thomas Brihaye, Véronique Bruyère, Aline Goeminne, and Jean-François Raskin, *Constrained existence problem for weak subgame perfect equilibria with omega-regular boolean objectives*, CoRR **abs/1806.05544** (2018).

Thomas Brihaye, Véronique Bruyère, Noémie Meunier, and Jean-François Raskin, *Weak subgame perfect equilibria and their application to quantitative reachability*, CSL Proceedings, LIPIcs, vol. 41, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015, pp. 504–518.

Patricia Bouyer, Romain Brenguier, Nicolas Markey, and Michael Ummels, *Pure Nash equilibria in concurrent deterministic games*, Logical Methods in Comput. Sci. **11** (2015), no. 2.

- Roderick Bloem, Krishnendu Chatterjee, and Barbara Jobstmann, *Graph games and reactive synthesis*, Handbook of Model Checking. (Edmund M. Clarke, Thomas A. Henzinger, Helmut Veith, and Roderick Bloem, eds.), Springer, 2018, pp. 921–962.
- Thomas Brihaye, Julie De Pril, and Sven Schewe, *Multiplayer cost games with simple Nash equilibria*, LFCS Proceedings, Lecture Notes in Comput. Sci., vol. 7734, Springer, 2013, pp. 59–73.
- Catriel Beeri, *On the membership problem for functional and multivalued dependencies in relational databases*, ACM Trans. Database Syst. **5** (1980), no. 3.
- Udi Boker, Thomas A. Henzinger, and Jan Otop, *The target discounted-sum problem*, LICS Proceedings, IEEE Computer Society, 2015, pp. 750–761.
- Véronique Bruyère, Quentin Hautem, and Jean-François Raskin, *On the complexity of heterogeneous multidimensional games*, Proceedings

CONCUR 2016, LIPIcs, vol. 59, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016, pp. 11:1–11:15.

———, *Games with lexicographically ordered ω-regular objectives*, CoRR **abs/1707.05968** (2017).

Véronique Bruyère, Quentin Hautem, Mickael Randour, and Jean-François Raskin, *Energy and mean-payoff games*, Work in progress, 2019.

Véronique Bruyère, Noémie Meunier, and Jean-François Raskin, *Secure equilibria in weighted games*, CSL-LICS Proceedings, ACM, 2014, pp. 26:1–26:26.

Véronique Bruyère, *Computer aided synthesis: A game-theoretic approach*, DLT, LNCS, vol. 10396, Springer, 2017, pp. 3–35.

Henrik Björklund, Sven Sandberg, and Sergei G. Vorobyov, *Memoryless determinacy of parity and mean payoff games: a simple proof*, Theor. Comput. Sci. **310** (2004), no. 1-3, 365–378.

- Krishnendu Chatterjee, Laurent Doyen, Emmanuel Filiot, and Jean-François Raskin, *Doomsday equilibria for omega-regular games*, Inf. Comput. **254** (2017), 296–315.
- Krishnendu Chatterjee, Laurent Doyen, and Thomas A. Henzinger, *Quantitative languages*, ACM Trans. Comput. Log. **11** (2010).
- Krishnendu Chatterjee, Laurent Doyen, Thomas A. Henzinger, and Jean-François Raskin, *Generalized mean-payoff and energy games*, FSTTCS'10, LIPIcs, vol. 8, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2010, pp. 505–516.
- Krishnendu Chatterjee, Thomas A. Henzinger, and Marcin Jurdzinski, *Games with secure equilibria*, Theor. Comput. Sci. **365** (2006), 67–82.
- Krishnendu Chatterjee, Thomas A. Henzinger, and Nir Piterman, *Generalized parity games*, FoSSaCS Proceedings, Lecture Notes in Comput. Sci., vol. 4423, Springer, 2007, pp. 153–167.

- Cristian S. Calude, Sanjay Jain, Bakhadyr Khoussainov, Wei Li, and Frank Stephan, *Deciding parity games in quasipolynomial time*, Proceedings STOC 2017, ACM, 2017, pp. 252–263.
- Krishnendu Chatterjee, Mickael Randour, and Jean-François Raskin, *Strategy synthesis for multi-dimensional quantitative objectives*, Acta Inf. **51** (2014), no. 3-4, 129–163.
- Julie De Pril, János Flesch, Jeroen Kuipers, Gijs Schoenmakers, and Koos Vrieze, *Existence of secure equilibrium in multi-player games with perfect information*, MFCS Proceedings, Lecture Notes in Comput. Sci., vol. 8635, Springer, 2014, pp. 213–225.
- E. Allen Emerson and Charanjit S. Jutla, *Tree automata, mu-calculus and determinacy*, FOCS Proceedings, IEEE Comp. Soc., 1991, pp. 368–377.
- A. Ehrenfeucht and J. Mycielski, *Positional strategies for mean payoff games*, Int. Journal of Game Theory **8** (1979), 109–113.

- Nathanaël Fijalkow and Florian Horn, *Les jeux d'accessibilité généralisée*, Technique et Science Informatiques **32** (2013), no. 9-10, 931–949.
- Erich Grädel, Wolfgang Thomas, and Thomas Wilke (eds.), *Automata, logics, and infinite games: A guide to current research*, Lecture Notes in Comput. Sci., vol. 2500, Springer, 2002.
- Erich Grädel and Michael Ummels, *Solution Concepts and Algorithms for Infinite Multiplayer Games*, New Perspectives on Games and Interaction, vol. 4, Amsterdam University Press, 2008, pp. 151–178.
- Hugo Gimbert and Wieslaw Zielonka, *When can you play positionally?*, MFCS Proceedings, Lecture Notes in Comput. Sci., vol. 3153, Springer, 2004, pp. 686–697.
- Florian Horn, *Explicit Muller games are PTIME*, FSTTCS Proceedings, LIPIcs, vol. 2, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2008, pp. 235–243.

- Paul Hunter and Jean-François Raskin, *Quantitative games with interval objectives*, FSTTCS Proceedings, LIPIcs, vol. 29, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2014, pp. 365–377.
- Neil Immerman, *Number of quantifiers is better than number of tape cells*, J. Comput. Syst. Sci. **22** (1981), 384–406.
- Marcin Jurdzinski, *Deciding the winner in parity games is in UP ∩ co-UP*, Inf. Process. Lett. **68** (1998), no. 3, 119–124.
- Richard M Karp, *A characterization of the minimum cycle mean in a digraph*, Discrete Mathematics **23** (1978), 309–311.
- Miroslav Klimos, Kim G. Larsen, Filip Stefanak, and Jeppe Thaarup, *Nash equilibria in concurrent priced games*, LATA Proceedings, Lecture Notes in Comput. Sci., vol. 7183, Springer, 2012, pp. 363–376.
- Harold W. Kuhn, *Extensive games and the problem of information*, Classics in Game Theory (1953), 46–68.

- Donald A. Martin, *Borel determinacy*, Annals of Mathematics **102** (1975), 363–371.
- John F. Nash, *Equilibrium points in n-person games*, PNAS, vol. 36, National Academy of Sciences, 1950, pp. 48–49.
- A. Rényi, *Representations of real numbers and their ergodic properties*, Acta Mathematica Academiae Scientiarum Hungarica **8** (1957), no. 3-4, 477–493.
- Mickael Randour, *Automated synthesis of reliable and efficient systems through game theory: a case study*, vol. abs/1204.3283, 2012.
- Reinhard Selten, *Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit*, Zeitschrift für die gesamte Staatswissenschaft **121** (1965), 301–324 and 667–689.
- Eilon Solan and Nicolas Vieille, *Deterministic multi-player Dynkin games*, Journal of Mathematical Economics **39** (2003), 911–929.

Michael Ummels, *The complexity of Nash equilibria in infinite multiplayer games*, FoSSaCS Proceedings, Lecture Notes in Comput. Sci., vol. 4962, Springer, 2008, pp. 20–34.

Michael Ummels and Dominik Wojtczak, *The complexity of Nash equilibria in limit-average games*, CoRR **abs/1109.6220** (2011).

______, *The complexity of Nash equilibria in limit-average games*, CONCUR Proceedings, Lecture Notes in Comput. Sci., vol. 6901, Springer, 2011, pp. 482–496.

Yaron Velner, *Robust multidimensional mean-payoff games are undecidable*, FoSSaCS Proceedings, Lecture Notes in Comput. Sci., vol. 9034, Springer, 2015, pp. 312–327.

Wieslaw Zielonka, *Infinite games on finitely coloured graphs with applications to automata on infinite trees*, Theor. Comput. Sci. **200** (1998), no. 1-2, 135–183.

Uri Zwick and Mike Paterson, *The complexity of mean payoff games on graphs*, Theor. Comput. Sci. **158** (1996), 343–359.