Combinatorics of shuffle products
(or how to shuffle a deck of cards)

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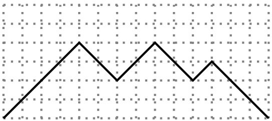
Enumerative Combinatorics

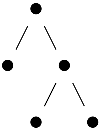
Enumerative problems arise in various contexts:

- discrete probability and statistical physics (compute probabilities in a discrete Markov chain)
- discrete geometry (counting integer points in polytopes)
- algebra and representation theory (count the multiplicity of an irreducible representation in a representation)
- many more examples

Counting problems

- **Exact formulas.** Dyck paths, binary trees:





$$\frac{1}{n+1}\binom{2n}{n}$$

- **Generating functions.** Alternating permutations such as 7162534,

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = z + 2\frac{z^3}{3!} + 16\frac{z^5}{5!} + 272\frac{z^7}{7!} + \ldots$$

- **Asymptotic formulas.** Integer partitions such as $9 = 4 + 3 + 1 + 1$.

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \text{ as } n \to \infty$$

- **Bijective problems**:
  Find bijections between sets with the same cardinality.
  Prove combinatorial identities by bijections, such as

$$\sum_{j=0}^{k} \binom{a}{j} \binom{b}{k-j} = \binom{a+b}{k}.$$

- **Structural problems**: partially ordered sets, group actions on set and related symmetries...

The computational side

- **Experimentation**: software such as SageMath can be used to manipulate combinatorial objects, make new conjectures, give evidence to old conjectures.
- **Proofs**: often the "generic" case of a proof is done by reasoning, leaving a finite number of cases to be checked by computer.
- **Algorithms**: some combinatorial construction have a strong algorithmic flavor.

Shuffle of a deck of cards

Definition

A *shuffle* of a sequence is done by:
1) splitting it in two parts,
2) create a new sequence containing the two parts, keeping their relative order.

Example

$$165482973 \to 1654 \mid 82973 \to 182697543$$

A permutation $\sigma_1 \dots \sigma_n$ of $1 \dots n$ is a shuffle of $1 \dots n$ if there is at most one i such that $i + 1$ is to its left. For example, 41256738.

The number of (nontrivial) shuffles of $1 \dots n$ is $2^n - n - 1$.

Perfect shuffles

Here we assume n is even. A *perfect shuffle* is when you split a deck in two equal parts, and combine the cards in an alternating way. It has two variants:

$$\pi_1 \;:\; 12345678 \to 15263748$$
$$\pi_2 \;:\; 12345678 \to 51627384.$$

Formally, π_i is in the symmetric group $\mathfrak{S}_n$, and a permutation σ acts on words by $\sigma \cdot (a_1 \dots a_n) = a_{\sigma^{-1}(1)} \dots a_{\sigma^{-1}(n)}$.

Perfect shuffles

Theorem (Elmsley)

You can move a chosen card i in top position of the deck in $\lfloor \log_2 n \rfloor$ operations, where each operation is π_1 or π_2.

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Suppose $n = 2^k$, number the cards from 0 to $n-1$, represent i by its binary expansion $a_1 \dots a_k$. Then the perfect shuffles are:

$$\pi_1 \ : a_1 \dots a_k \to a_2 \dots a_k a_1$$
$$\pi_2 \ : a_1 \dots a_k \to a_2 \dots a_k \overline{a_1}$$

($\overline{a_1} = 1 - a_1$). You can use this to get $0 \dots 0$ in k steps.

Perfect shuffles

Diaconis, Graham, Kantor (1983) computed the group generated by the perfect shuffles π_1 and π_2.

When $n = 24$, the answer involves one of the sporadic finite simple groups, the *Mathieu group* M_{12}.

They relate perfect shuffle with parallel computing and an $O(\log n)$ fast Fourier transform algorithm.

Let ω_n denote the order of π_1 when there are $2n$ cards. This is the order of 2 in the ring of integers modulo $2n - 1$. Very little is known about this sequence, number theory is involved.

Perfect shuffles

There exists other types of perfect shuffles. The *Monge perfect shuffle* is done by reversing one set of cards before mixing the two sets:

$$12345678 \mapsto 18273645$$

Cf. Lachal 2010: computations of the periods of this shuffles (and its variants) via arithmetic.

Riffle shuffle

A *riffle shuffle* is done by chosing uniformly one shuffle among the $2^n - n - 1$ shuffles of $1, \ldots, n$, and permute the deck of cards accordingly.

Remark

There are effective ways to describe this operation. Begin by splitting the deck of n cards in two sets, according to a binomial distribution: the probability to get sets of size k and $n-k$ is $\binom{n}{k}\frac{1}{2^n}$.

Then choose uniformly a k-element subset of $1 \ldots n$ which will give the positions of cards in the first set.

(To avoid trivial shuffles... repeat the operation until you get a nontrivial shuffle !)

Riffle shuffle

Even more, there is a practical way to choose a random subset of size k among the $\binom{n}{k}$ choices.

At each step, there are i (resp. j) cards remaining cards in the first set (resp. second set). Then the next card you pick is from set 1 with probability $i/(i+j)$ and from set 2 with probability $j/(i+j)$.

Start with $(i,j) = (k, n-k)$ and finish when $i = j = 0$.

Riffle shuffle

Problem

Take a deck 52 cards, perfectly sorted.
How many shuffles do you need to perform to get a randomly sorted deck ?

Theorem

In a "human" situation, 7 is more than enough.

Bayer and Diaconis (1992), Trefethen (2000).

The formalization of the problem comes from:
Gilbert-Shannon-Reeds (1955), Diaconis (1988).

It leads to consider a Markov chain on the symmetric group $\mathfrak{S}_{52}$.

Riffle shuffle

Given a sequence $(a_1, \ldots, a_n)$, and a permutation $\sigma \in \mathfrak{S}_n$, the action of σ is

$$\sigma \cdot (a_1, \ldots, a_n) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}).$$

Definition

A *descent of a permutation* $\sigma \in \mathfrak{S}_n$ is an index $1 \leq i \leq n-1$ such that $\sigma(i) > \sigma(i+1)$.

Lemma

A shuffle is the action of a permutation $\sigma \in \mathfrak{S}_n$ with only one descent.

For example, $147823569 \cdot 165482973 = 182697543$.

Riffle shuffle

Remark

A permutation $\sigma_1 \dots \sigma_n$ can be seen in two different ways:

- it is a deck of cards (upon numbering cards from 1 to n),
- it acts on deck of cards by permuting cards.

The second point of view is natural to compose permutations. But we want to avoid using the huge group $\mathfrak{S}_{n!}$.

We need to identify the permutation σ with the "translation" $\tau \mapsto \tau\sigma^{-1}$.

The group algebra

It is convenient to work in the *group algebra* $\mathbb{Z}[\mathfrak{S}_n]$. Its elements are formal sums of permutations with integers coefficients.

Remark

An element $\sum_{\sigma \in \mathfrak{S}_n} a_\sigma \sigma$ where $a_\sigma \geq 0$ naturally gives a probability distribution on $\mathfrak{S}_n$ by:

$$\mathbb{P}(\sigma) = \frac{a_\sigma}{\sum a_\sigma}.$$

Think of a_σ as a “non-normalized” probability.

Consider the sum of all shuffles:

$$E_1 = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma \text{ has 1 descent}}} \sigma.$$

The group algebra

Proposition

Consider the expansion

$$E_1^k = \sum_{\sigma \in \mathfrak{S}_n} A_{k,\sigma} \sigma.$$

then $A_{k,\sigma}$ is the number of ways to get σ from $1, 2, 3, \ldots, n$ after k shuffles.

So $A_{k,\sigma}/(\sum A_{k,\sigma})$ is the probability to get σ after k (uniformly chosen) shuffles applied to $123\ldots n$.

Proof.

By definition, $A_{k,\sigma}$ is the number of factorizations $\sigma = \sigma_1 \cdots \sigma_k$ where each σ_i has 1 descent. And $\sum A_{k,\sigma}$ is the number of k-tuples of permutations with 1 descent.

The Eulerian algebra

Theorem (Loday, 1994)

The elements

$$E_k = \sum_{\sigma \in \mathfrak{S}_n,\ k \text{ descents}} \sigma \qquad \text{for } 0 \leq k \leq n-1,$$

linearly span a n-dimensional subalgebra of $\mathbb{Z}[\mathfrak{S}_n]$.

It is called the *descent algebra*. This means there is an expansion $E_i E_j = \sum_k c_k E_k$.

So computing E_1^k can be done in a n-dimensional vector space !

The Eulerian subalgebra

This algebra is named after the *Eulerian numbers*. They are integers $A_{n,k}$ counting the number of permutations in $\mathfrak{S}_n$ with k descents. In particular $A_{n,k}$ is the number of terms in the sum E_k.

Generating function: $\sum_{k,n \geq 0} A_{n,k} z^n t^k = \frac{t-1}{t-e^{(t-1)z}}$.

$$
\begin{array}{lllll}
1 & & & & \\
1 & 1 & & & \\
1 & 4 & 1 & & \\
1 & 11 & 11 & 1 & \\
1 & 26 & 66 & 26 & 1 \\
\vdots & & \vdots & & \vdots
\end{array}
$$

Idempotents

Theorem

The Eulerian algebra has a basisof orthogonal idempotents, i.e. a linear basis $(P_i)_{1\leq i\leq n}$ *such that* $P_iP_j = \delta_{i,j}P_i$.

One of the idempotent is $\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sigma$.
It represents the uniform probability distribution on $\mathfrak{S}_n$.

If $E_1 = \sum a_iP_i$ then $E_1^k = \sum a_i^kP_i$. We can get the rate of convergence to the uniform distribution !

Some bijective problems, coming from the Eulerian algebra:

If σ, τ have the same number of descents, find a bijection between:

- factorizations $\sigma = \alpha\beta$ where $\mathrm{des}(\alpha) = i$, $\mathrm{des}(\beta) = j$, and
- factorizations $\tau = \alpha\beta$ where $\mathrm{des}(\alpha) = i$, $\mathrm{des}(\beta) = j$.

(This proves the existence of the algebra.)

For each $\sigma \in \mathfrak{S}_n$, find a bijection between:

- factorizations $\sigma = \alpha\beta$ where $\mathrm{des}(\alpha) = i$, $\mathrm{des}(\beta) = j$, and
- factorizations $\sigma = \alpha\beta$ where $\mathrm{des}(\alpha) = j$, $\mathrm{des}(\beta) = i$.

(This shows the commutativity.)

`Thanks for your attention.`